Double Barrier Option Pricing for Double Exponential Jump Diffusion Model

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Abstract

In the setting of double exponential jump diffusion model, I present the analytical solution for two-dimensional Laplace transform of double barrier option prices. The two-dimensional inversion algorithm of Laplace transform uses the techniques developed by Choudhury, Lucantoni and Whitt (1994). The numerical results of the double barrier option prices indicate that the method is fast, accurate, and easy to implement without requiring high precision calculations in Laplace inversion.
Keywords : jump diffusion, double barrier option and Laplace transform

摘 要

本文給出了在雙指數跳動擴散模型下, 雙障礙期權價格的拉普拉斯變換的解析解. 我們利用 Choudhury, Lucantoni 和 Whitt (1994) 給出的二維拉普拉斯逆變換算法得到了期權的價格。雙障礙期權價格的數值結果表明了算法是快速, 準確, 易于實行的。

關鍵詞: 跳動擴散, 雙障礙期權和拉普拉斯變換

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Chapter 1

Introduction

Option, the right to buy or sell an asset on a certain day at a specified price, has existed for a long time. Its record dated back to the ancient Greek mathematician and philosopher Thales, who used options to secure a low price for olive presses in advance of the harvest. Despite the long history that options had existed, standardized, listed call options on stocks were not traded until the Chicago Board of Option Exchange (CBOE) began the trading in April of 1973. Since 1975, a number of other exchanges have got into the game, including the American Stock Exchange (AMEX), the Pacific Stock Exchange (PSE) (now the Philadelphia Stock Exchange (PHE)), the International Securities Exchange (ISE), Boston Options Exchange (BOX), and Archipelago (now NYSE Arca). Today, different types of options are traded in the financial market. For example, a European option may only be exercised at maturity. An American option may be exercised on any trading day on or before maturity. A barrier option is the option with the general characteristic that the underlying security's price must reach some trigger level before the exercise can occur.

The value of an option can be estimated using a variety of quantitative techniques based on the concept of risk neutral pricing and using stochastic calculus. Among the theoretical models, the most basic one is the Black-Scholes-Merton model, which is based on Brownian motion and normal distribution. Option pricing papers within this classical model that are particularly relevant to the current paper are: Black and Scholes (1973) model for the European call and put options; Black (1976) model for options on futures contracts; Heather, Jarrow and Morton (1992) model for options on bonds; and Brace, Gatarek, and Musiela model (1997) for caps and floors.

Despite the model's popularity which is due to its simplicity of using observable parameters and easy calculation, Kou (2002) points out that the model is trapped in two puzzles emerged from many empirical investigations.

1. The leptokurtic and asymmetric features. In the classic Black-Scholes-Merton model, the marginal distribution of the underlying assets is assumed to be normal. However, many empirical studies suggest that the distribution is skewed to the left, and has a higher peak and two heavier tails than those of the normal distribution.

2. The volatility smile. More precisely, if the Black-Scholes-Merton model is correct, then the implied volatility should be constant; but it is widely recognized that the implied volatility resembles a "smile", when it is plotted against the increasing strike price.

Aiming at explaining the two puzzles, an explosion of new models have been introduced to modify the Black-Scholes-Merton model to incorporate the leptokurtic feature and the "volatility smile". Popular ones are: (a) the jump diffusion models of Merton (1976), Bates (1991) and Madan and Chang (1996); (b) the stochastic volatility models of Heston (1993), Hull and White (1987), Melino and Turnbull (1990) and Stein and Stein (1991); (c) the constant-elasticity-of-variance (CEV) models of Cox and Ross (1976) and Davydov and Linetsky (2001).

The goal of this thesis is to price a double barrier option from the perspective of the jump diffusion model. A double barrier option, by definition, has two barriers, one above and the other below the current stock. It is considered as a path dependent option since the payoff to the holder depends on the breaching behaviors of the stock price process with respect to the two barriers. A double barrier option contract specifies three kinds of payoffs, which are dependent on whether the up-barrier or down-barrier is hit, or no breaching of either barrier throughout the whole life of the option. A barrier is said to be of knock-out type if the resulting payoff upon hitting by the stock price is a rebate payment. A great number of double barrier options can be designed to achieve a wide variety of risk management functions through various structures. Like single barrier options, an investor buying a double barrier option may use the more exotic forms of the double barrier feature to achieve reduction in option premium, match investor's belief about the future movement of the stock price process and/or match his specific hedging needs.

To price the double barrier option, I apply the double exponential jump diffusion model proposed by Kou (2002). This model consists of a continuous part driven by a Brownian motion and a jump part with jump size having a double exponential distribution. Besides, as in Kou (2002), the model incorporates the following properties.

1. It has the leptokurtic and asymmetric features, under which the return distribution of the assets has a higher peak and two heavier tails than the normal distribution, especially the left tail;

2. It can reproduce the "volatility smile";

3. It has the analytical tractability due to the memoryless property of the exponential distribution.

In the thesis, I derive the "closed form" pricing formulae for double barrier option via Laplace transform. To execute the Laplace inversion, I use the two-dimensional inversion algorithm developed by Choudhury, Lucantoni and Whitt (1994). The numerical results of the double barrier option prices indicate that the method is fast, accurate, and easy to implement without requiring high precision calculations in Laplace inversion.

The rest of the thesis is organized as follows: In Chapter 2, I begin to review some important models in option pricing. As a highlight, I review the double exponential jump diffusion model and the inversion algorithm of the Laplace transform as well as the Monte Carlo method for the double exponential jump diffusion process. In Chapter 3 the Laplace transforms of the first passage time of the double barrier option price are given. Taking advantage of those transforms, I derive the closed-form solution of the two-dimensional Laplace transform of the double barrier option price. In Chapter 4 numerical results of the prices are presented, while the last chapter is the conclusion.

Chapter 2

Literature Review

In this chapter, I go over some important models under which option pricing problems are studied. As a highlight, I introduce Kou's double exponential jump diffusion model (2002) under which the price of double barrier options will be studied in this thesis. Later in this chapter, Laplace transform inversion algorithm and Monte Carlo method for double exponential jump diffusion processes are also included.

2.1 Review of the Models

2.1.1 Black-Scholes-Merton Model

The Black-Scholes-Merton model was the first quantitative technique to comprehensively and accurately estimate the price for a variety of simple option contracts. The key assumptions of the Black-Scholes-Merton model are:

- The price of the underlying instrument S_t follows a geometric Brownian motion with constant drift μ and volatility σ, and the price changes are log-normally distributed:

$$dS_t = \mu S_t dt + \sigma S_t dW_t.$$

- It is possible to short sell the underlying stock.
- There are no arbitrage opportunities.
- Trading in the stock is continuous.
- There are no transaction costs or taxes.
- All securities are perfectly divisible (e.g. it is possible to buy any fraction of a share).
- It is possible to borrow and lend cash at a constant risk-free interest rate.
- The stock does not pay a dividend.

The above lead to the following formula for the price C of a European call option with exercise price K on a stock currently trading at price S, i.e., the right to buy a share of the stock at price K after T years. The constant interest rate is r.

$$C(S,T) = S\Phi(d_1) - Ke^{-rT}\Phi(d_2),$$

where

$$
\begin{aligned}
d_1 &= \frac{\ln(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, \\
d_2 &= \frac{\ln(S/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}.
\end{aligned}
$$

Here Φ is the standard normal cumulative distribution function.

Since then, option pricing papers within this classical model that are particularly relevant to the current paper have included: Black (1976) model for options on futures contracts; Heather, Jarrow and Morton (1992) model for options on bonds; and Brace, Gatarek, and Musiela model (1997) for caps and floors.

German and Yor (1996) prices the double barrier option in the standard Black and Scholes framework. The paper uses the standard risk-neutral valuation techniques and the Markov property as well as the Cameron-Martin theorem to obtain a simple expression of the Laplace transform of the price of the double barrier option with fixed boundaries. Denoting $\mathcal{L}^{-1}$ for the inversion of the Laplace transform operator, the double knock-out call price at time t has the form

$$
C_{L,U}(t) = S_t\{C(0, 1, \sigma, T-t, K/S_t) - e^{-r(T-t)}\{\mathcal{L}^{-1}\psi\}(T-t)\},
$$

where L and U are the lower and the upper boundaries, respectively, K is the strike price, T is the maturity, r is the risk-free interest rate, S_t is the stock price at time t, $C(0, 1, \sigma, T-t, K/S_t)$ is the Black and Scholes price of a standard call with maturity $T-t$, strike price K/S_t, an underlying asset with initial price at 1, and $\psi(\lambda) = \frac{1}{\sigma^2}\phi(\frac{\lambda}{\sigma^2})$ with

$$
\begin{aligned}
\phi(\theta) &= \frac{\mathrm{sh}(yb)}{\mathrm{sh}(y(a+b))} g(e^{-a})e^{-\nu a} + \frac{\mathrm{sh}(ya)}{\mathrm{sh}(y(a+b))} g(e^{b})e^{\nu b}, \\
g(e^z) &= \frac{e^{-\nu z}}{y} \int_{\ln h}^{\infty} dx (e^{-y|z-x|})[e^{(\nu+1)x} - he^{\nu x}],
\end{aligned}
$$

$$\begin{aligned}\nu &= \frac{1}{\sigma^2}(\mu - \frac{\sigma^2}{2}), \ \ \frac{y^2}{2} = \theta + \frac{\nu^2}{2}, \\ e^{-a} &= \frac{L}{S_t}, \ \ e^b = \frac{U}{S_t} \text{ and } h = \frac{K}{S_t}.\end{aligned}$$

Using the methodology developed in Geman-Eydeland (1995), the paper gets the price of the double barrier option and compares the result with the Monte Carlo simulation. The consequence is that the inversion of the Laplace transform requires an order of magnitude less operations than the Monte Carlo simulation.

Although the calculation of many financial instruments is easy under the Black-Scholes-Merton model, as mentioned before, the model cannot give explanations on empirical evidence, such as the leptokurtic feature and the "volatility smile".

2.1.2 Merton's Jump Diffusion Model

Merton (1976) generalizes the pure jump process and introduces a model which posits its continuous part to be a Wiener process and the jump component to be a "Poisson-driven" process. Accordingly, the stock price returns are a mixture of both types and can be formally written as a stochastic differential equation:

$$\frac{dS_t}{S_t} = (\alpha - \lambda k)dt + \sigma dZ + dq,$$

where α is a instantaneous expected return on the stock; σ^2 is the instantaneous variance of the return, conditional on no arrivals of important new information (i.e., the Poisson event does not occur.); and dZ is a standard Gauss-Wiener process. $q(t)$ is the independent Poisson process. dq and dZ are assumed to be independent. λ is the mean number of arrivals per unit time. $k \equiv \epsilon(Y-1)$ where $(Y-1)$ is the random variable percentage change in the stock price if the Poisson event occurs and ϵ is the expectation operator over the random variable Y.

Assume that Y has a lognormal distribution. Let δ^2 denote the variance of the logarithm of Y, and suppose that the expected value of Y is 1. If S is the current stock price and τ is the remaining time to expire, then the price of a European call with a strike price E is

$$F(S,\tau) = \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} W(S,\tau;E,\sigma^2+(n/\tau)\delta^2,r).$$

Here $W(S,\tau;E,\sigma^2+(n/\tau)\delta^2,r)$ is the Black-Scholes option pricing formula for the no-jump case. From the formula, W can be written as:

$$W(S,\tau;E,\sigma^2+(n/\tau)\delta^2,r) = S\Phi(d_1) - Ee^{-rT}\Phi(d_2),$$

where

$$d_1 = \frac{\ln(S/E) + (r + (\sigma^2 + (n/\tau)\delta^2)^2/2)T}{(\sigma^2 + (n/\tau)\delta^2)\sqrt{T}},$$
$$d_2 = \frac{\ln(S/E) + (r - (\sigma^2 + (n/\tau)\delta^2)^2/2)T}{(\sigma^2 + (n/\tau)\delta^2)\sqrt{T}}.$$

Here Φ is the standard normal cumulative distribution function.

As a result, Merton's model can lead to leptokurtic feature, implied volatility smile and analytical solutions for European call and put options and interest rate derivatives. However, the normal jump diffusion model is lack of the analytical tractability for the path-dependent options.

The Merton type jump diffusion model has been widely used in academic research. For example, Ball and Torous (1983) put forth a Bernoulli jump process for common stock returns to do statistical estimation and empirical analysis; Bates (1991) uses the model to examine for evidence of expectations prior to October 1987 of an impending stock market crash.

2.1.3 Stochastic Volatility Jump Diffusion Model

Although the Black-Scholes formula is often quite successful in explaining stock option prices, it does have known bias. This is because the Black-Scholes model makes the strong assumption that stock returns are normally distributed with known mean and variance. Since the Black-Scholes formula does not depend on the mean spot return, it cannot be generalized by allowing the mean to vary. But the variance assumption is somewhat dubious. Hence, Heston (1993) introduces a model of stochastic volatility which can be written as

$$dS(t) \;=\; \mu S dt + \sqrt{v(t)} S dz_1(t),$$

where $z_1(t)$ is a Wiener process. The volatility follows the process

$$dv(t) \;=\; \kappa[\theta - v(t)]dt + \sigma\sqrt{v(t)} dz_2(t),$$

where $z_2(t)$ has correlation ρ with $z_1(t)$.

Given the spot stock price S and volatility v and using the characteristic function method, the European call option with strike price K and maturity T has the analytical solution

$$C(S,v,t) \;=\; SP_1 - Ke^{-r(T-t)}P_2,$$

where

$$P_j(x,v,t;\ln[K]) \;=\; \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}[\frac{e^{-i\phi\ln[K]}f_j(x,v,t;\phi)}{i\phi}]d\phi, \quad j = 1,2,$$

and

$$\begin{aligned} x \;&=\; \ln S, \;\; u_1 \;=\; \frac{1}{2}, \;\; u_2 \;=\; -\frac{1}{2}, \\ a \;&=\; \kappa\theta, \;\; b_1 \;=\; \kappa + \lambda - \rho\sigma, \;\; b_2 \;=\; \kappa + \lambda, \end{aligned}$$

$$
\begin{aligned}
f_j(x, v, t; \phi) &= e^{C(T-t;\phi)+D(T-t;\phi)v+i\phi x}, \quad j = 1, 2, \\
C(\tau; \phi) &= r\phi i\tau + \frac{a}{\sigma^2}(b_j - \rho\sigma\phi i + d)\tau - 2\ln[\frac{1-ge^{d\tau}}{1-g}], \\
D(\tau; \phi) &= \frac{b_j - \rho\sigma\phi i + d}{\sigma^2}[\frac{1-e^{d\tau}}{1-ge^{d\tau}}], \\
g &= \frac{b_j - \rho\sigma\phi i + d}{b_j - \rho\sigma\phi i - d}, \\
d &= \sqrt{(b_j - \rho\sigma\phi i)^2 - \sigma^2(2u_j\phi i - \phi^2)}.
\end{aligned}
$$

With this model, Heston also provides the closed-form solution for bond options and currency options. The correlation between volatility and the spot asset's price is important for explaining return skewness. The kurtosis is changed by stochastic volatility. Besides, this model can also reproduce "volatility smile". The main deficiency of Heston's model is that it is lack of analytical tractability, especially for path-dependent options and complex interest rate derivatives.

The stochastic volatility model has acquired a lot of attention in academic research. For example, Hull and White (1987) derives a series solution for the price of a European call option on a security with a stochastic volatility that is uncorrelated with the security price. The case in which the volatility is correlated with the stock price is examined using numerical methods. Melino and Turnbull (1990) investigates the consequences of stochastic volatility for pricing spot foreign currency options. Stein and Stein (1991) studies the stock price distributions that arise when prices follow a diffusion process with a stochastically varying volatility parameter. Using analytical techniques, it derives an explicit closed-form solutions for the case where volatility is driven by the Ornstein-Uhlenbeck process.

In the paper of Duffie, Pan and Singleton (2002), jumps are incorporated into the stochastic volatility model. The new model which called the affine jump-diffusion (AJD) is, roughly speaking, a jump-diffusion process for which the drift vector, "instantaneous" covariance matrix, and jump intensities all have affine dependence on the state vector. Assuming that the state vector X is a Markov process in some state space $D \subset \mathbb{R}^n$, the affine jump-diffusion can mathematically be given by the equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t + dZ_t,$$

where W is a standard Brownian motion in $\mathbb{R}^n$; $\mu : D \to \mathbb{R}^n, \sigma : D \to \mathbb{R}^{n\times n}$, and Z is a pure jump process whose jumps have a fixed probability distribution ν on $\mathbb{R}^n$ and arrive with intensity $\{\lambda(X_t) : t \geq 0\}$, for some $\lambda : D \to [0, \infty)$.

Fix an affine discount-rate function $R : D \to \mathbb{R}$. The affine dependence of $\mu, \sigma\sigma^\top, \lambda$, and R are determined by coefficients (K, H, l, ρ) defined by:

- $\mu(x) = K_0 + K_1 x$, for $K = (K_0, K_1) \in \mathbb{R}^n \times \mathbb{R}^{n\times n}$.
- $(\sigma(x)\sigma(x)^\top)_{ij} = (H_0)_{ij} + (H_1)_{ij} \cdot x$, for $H = (H_0, H_1) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n\times n}$.
- $\lambda(x) = l_0 + l_1 \cdot x$, for $l = (l_0, l_1) \in \mathbb{R} \times \mathbb{R}^n$.
- $R(x) = \rho_0 + \rho_1 \cdot x$, for $\rho = (\rho_0, \rho_1) \in \mathbb{R} \times \mathbb{R}^n$.

By deriving a closed-form expression for an extended transform of the AJD process X, the paper shows that this transform leads to analytical tractable pricing relations for a wide variety of valuation problems. With regard to option pricing, fixing the current date 0 and future expiration date T, we suppose that the generalized terminal payoff function $(e^{d \cdot X(T)} - c)^+$. Hence, the price of a European call has the closed form

$$\begin{aligned} C(d, c, T) &= E(\exp(-\int_0^T R(X_s)ds)(e^{d \cdot X(T)} - c)^+) \\ &= G_{d,-d}(-\ln c; X_0, T) - cG_{0,-d}(-\ln c; X_0, T), \end{aligned}$$

where, given some $(x, T, a, b) \in D \times [0, \infty) \times \mathbb{R} \times \mathbb{R}$, $G_{a,b}(.; x, T) : \mathbb{R} \to \mathbb{R}_+$ is given by

$$G_{a,b}(y; X_0, T) = \frac{\psi(a, X_0, 0, T)}{2} - \frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}[\psi(a + ivb, X_0, 0, T)e^{-ivy}]}{v} dv.$$

Here

$$\psi(u, x, t, T) = e^{\alpha(t)+\beta(t)\cdot x},$$

where β and α satisfy the complex-valued ODEs

$$\begin{aligned}
\dot{\beta}_t &= \rho_1 - K_1^\top \beta_t - \frac{1}{2}\beta_t^\top H_1 \beta_t - l_1(\theta(\beta_t) - 1), \\
\dot{\alpha}_t &= \rho_0 - K_0 \cdot \beta_t - \frac{1}{2}\beta_t^\top H_0 \beta_t - l_0(\theta(\beta_t) - 1),
\end{aligned}$$

with boundary conditions $\beta(T) = u$ and $\alpha(T) = 0$. θ is well defined by $\theta(p) = \int_{\mathbb{R}^n} \exp(p \cdot z) d\nu(z)$.

The main disadvantage of the AJD model lies in the difficulties leading to analytical solutions for path-dependent options.

2.1.4 Constant Elasticity of Variance (CEV) Model

First introduced by Cox and Ross (1976), the constant elasticity of variance (CEV) model assumes that the volatility is a function of the underlying asset. Compared to the Black-Scholes model, the CEV model provides a better fit to the empirical observations. Mathematically, it can be written as

$$dS = \mu S_t dt + \delta S_t^{\beta+1} dB_t, \quad t \geq 0, \quad S_0 = S > 0,$$

where $\{B_t, t \geq 0\}$ is a standard Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0})$, μ is a constant ($\mu = r - q$, where $r \geq 0$ and $q \geq 0$ are the constant risk-free interest rate and the constant dividend yield, respectively), and δS_t^β is a given local volatility function.

Davydov and Linetsky (2001) studies the pricing and hedging of barrier and look-back options under the CEV process given the initial price $S_0 = S$. For the double knock-out call, we suppose that the strike price is K, the maturity is T, the the lower and upper boundaries are L and U respectively and the first exit time from an interval between the two barriers is $\tau_{\{L,U\}}$, the Laplace transform of the option price is

$$\int_0^\infty e^{-\lambda T} E[1_{\{\tau_{\{L,U\}}>T\}}(S_T-K)^+]dT = \frac{1}{\omega_\lambda \Delta_\lambda(L,U)}$$

$$\times \begin{cases} \Delta_\lambda(L,S)[\psi_\lambda(U)J_\lambda(K,K,U)-\phi_\lambda(U)I_\lambda(K,K,U)], & \text{if } S \le K, \\ \Delta_\lambda(L,S)[\psi_\lambda(U)J_\lambda(K,S,U)-\phi_\lambda(U)I_\lambda(K,S,U)] & \\ +\Delta_\lambda(S,U)[\phi_\lambda(L)I_\lambda(K,K,S)-\psi_\lambda(L)J_\lambda(K,K,S)], & \text{if } S > K, \end{cases}$$

where

$$\begin{aligned}
I_\lambda(K,A,B) &:= \int_A^B (Y-K)\psi_\lambda(Y)m(Y)dY, \\
J_\lambda(K,A,B) &:= \int_A^B (Y-K)\phi_\lambda(Y)m(Y)dY, \\
\Delta_\lambda(A,B) &= \phi_\lambda(A)\psi_\lambda(B)-\psi_\lambda(A)\phi_\lambda(B), \\
m(S) &= \frac{2}{\delta^2 S^{2\beta+2}\Im(S)}, \\
\Im(S) &= \exp(\frac{\mu}{\delta^2\beta}S^{-2\beta}), \\
\omega_\lambda &= \begin{cases} \frac{2|\mu|\Gamma(2m+1)}{\delta^2\Gamma(m-k+1/2)}, & \mu \neq 0, \\ |\beta|, & \mu = 0, \end{cases} \\
\psi_\lambda(S) &= \begin{cases} S^{\beta+\frac{1}{2}}e^{\frac{\epsilon}{2}x}M_{k,m}(x), & \beta<0,\ \mu\neq 0, \\ S^{\beta+\frac{1}{2}}e^{\frac{\epsilon}{2}x}W_{k,m}(x), & \beta>0,\ \mu\neq 0, \\ S^{\frac{1}{2}}I_\nu(\sqrt{2\lambda}z), & \beta<0,\ \mu=0, \\ S^{\frac{1}{2}}K_\nu(\sqrt{2\lambda}z), & \beta>0,\ \mu=0, \end{cases} \\
\phi_\lambda(S) &= \begin{cases} S^{\beta+\frac{1}{2}}e^{\frac{\epsilon}{2}x}W_{k,m}(x), & \beta<0,\ \mu\neq 0, \\ S^{\beta+\frac{1}{2}}e^{\frac{\epsilon}{2}x}M_{k,m}(x), & \beta>0,\ \mu\neq 0, \\ S^{\frac{1}{2}}K_\nu(\sqrt{2\lambda}z), & \beta<0,\ \mu=0, \\ S^{\frac{1}{2}}I_\nu(\sqrt{2\lambda}z), & \beta>0,\ \mu=0, \end{cases} \\
x &:= \frac{|\mu|}{\delta^2|\beta|}S^{-2\beta},\quad z := \frac{1}{\delta|\beta|}S^{-\beta},
\end{aligned}$$

$$\begin{aligned} m &:= \frac{1}{4|\beta|}, \quad \epsilon = \mathrm{sign}(\mu\beta), \\ k &:= \epsilon(\frac{1}{2} + \frac{1}{4\beta}) - \frac{\lambda}{2|\mu\beta|}, \quad \nu := \frac{1}{2|\beta|}. \end{aligned}$$

Here $M_{k,m}(x)$ and $W_{k,m}(x)$ are the Whittaker functions, and $I_\nu(x)$ and $K_\nu(x)$ are the modified Bessel functions and $\Gamma(x)$ is the Euler Gamma function.

For the scenario of the standard lookback option, let M_t and m_t be the maximum and minimum prices recorded to date t. Since the probability distributions of the maximum and minimum are recovered by inverting the Laplace transforms, lookback prices are expressed in terms of these probabilities. More precisely, the prices of the standard lookback call and the standard lookback put are

$$\begin{aligned} e^{-r(T-t)}E_t[(S_T - m_T)^+] &= e^{-q(T-t)}S_t - e^{-r(T-t)}m_t + e^{-r(T-t)}\int_0^{m_t} F(y; S_t, T-t)dy, \\ e^{-r(T-t)}E_t[(M_T - S_T)^+] &= e^{-r(T-t)}M_t - e^{-q(T-t)}S_t + e^{-r(T-t)}\int_{M_t}^{\infty} G(y; S_t, T-t)dy, \end{aligned}$$

where

$$\begin{aligned} \int_0^\infty e^{-\lambda t}F(y; x, t)dt &= \frac{1}{\lambda}\frac{\phi_\lambda(x)}{\phi_\lambda(y)}, \\ \int_0^\infty e^{-\lambda t}G(y; x, t)dt &= \frac{1}{\lambda}\frac{\psi_\lambda(x)}{\psi_\lambda(y)}. \end{aligned}$$

Here $\phi_\lambda(x)$ and $\psi_\lambda(x)$ are the same functions as before.

The contribution of this study are two-fold. First, it derives pricing formulae for barrier and lookback options under the CEV process in closed form. The analytical formulae allow fast and accurate calculation of prices and hedge ratios of barrier and lookback options under the CEV process on a PC. Second, it applies the analytical formulae to carry out a comparative statics analysis and demonstrate that, in a presence of a CEV-based volatility smile, barrier and lookback option prices and their hedge ratios can deviate dramatically from the lognormal values. In particular, up-and-out, double knock-out, and lookback call prices and deltas are extremely sensitive to the specification of the elasticity parameter β.

Besides Davydov and Linetsky's work, there is some other work under the CEV model. For example, Anderson and Anderson (2000) leads to the analytical solution for interest rate derivatives. However, the CEV model does not have the leptokurtic feature. More precisely, the return distribution in the CEV model has a thinner right tail than that of the normal distribution. Moreover, the implied volatility can only be a monotone function of the strike price. Therefore, the CEV model cannot reproduce the "volatility smile".

2.2 Kou's Double Exponential Jump Diffusion Model

2.2.1 The Model Formulation

The double exponential jump diffusion model assumes that the return process has two components part modeled as Brownian motion, and a jump part with jumps having a double exponential distribution and with jump times driven by a Poisson process. It is shown (Kou, 2002) that under such a model, when using a HARA type utility function, the rational-expectations equilibrium price of an option is given by the expectation of the discounted option payoff under a risk-neutral measure P^*. Under P^*, the dynamics of the asset price $S(t)$ is given by

$$\frac{dS(t)}{S(t-)} = (r - \lambda\zeta)dt + \sigma dW(t) + d(\sum_{i=1}^{N(t)}(V_i - 1)), \tag{2.1}$$

with the return process $X(t) = \log(S(t)/S(0))$ given by

$$X(t) = (r - \frac{1}{2}\sigma^2 - \lambda\zeta)t + \sigma W(t) + \sum_{i=1}^{N(t)} Y_i, \quad X(0) = 0. \tag{2.2}$$

Here $W(t)$ is a standard Brownian motion, $N(t)$ a Poisson process with rate λ, and V_i a sequence of independent identically distributed nonnegative random variables such that $Y = \log(V)$ has asymmetric double exponential distribution with the density

$$f_Y(y) \quad = \quad p \cdot \eta_1 e^{-\eta_1 y} 1_{\{y \geq 0\}} + q \cdot \eta_2 e^{\eta_2 y} 1_{\{y<0\}}, \eta_1 > 1, \eta_2 > 0,$$

where $p, q \geq 0, p+q = 1$. Here the condition $\eta_1 > 1$ is imposed to ensure that the asset price $S(t)$ has finite expectation. The constant ζ is given by

$$\zeta = E[V_i] - 1 = \frac{p\eta_1}{\eta_1 - 1} + \frac{q\eta_2}{\eta_2 + 1} - 1.$$

2.2.2 The Merits of the Model

In the paper of Kou (2002), the author shows that the double exponential jump diffusion model has the following merits when compared with other models:

1. The model is internally self-consistent. In the category of finance, it means that the model is arbitrage-free and can be embedded in an equilibrium setting which has been shown in (2.1).

2. The model can give explanation to some important empirical investigation. It can incorporate two empirical features: (1) The asymmetric leptokurtic features, which can be characterized that the return distribution of the assets has a higher peak and two heavier tails than the normal distribution. Figure 1 is taken from Kou (2002) to demonstrate the leptokurtic feature of the model. The first panel compares the overall shapes of the density of the double exponential jump diffusion and the normal density with the same mean and variance, the second one details the shape around the peak area, and the last two show the left and right tails. (The dotted line is used for the normal density, while the solid one is used

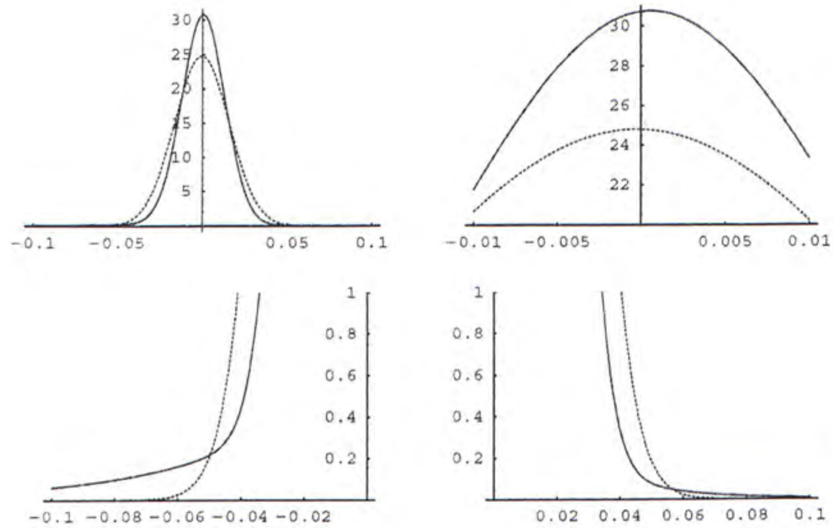

Figure 2.1: The Leptokurtic Feature

for the model.) (2) The volatility smile, which indicates that the implied volatility curve is convex and resembles a "smile", in contrast to the constant volatility if the Black-Scholes model is correct. Also taken from Kou (2002), Figure 2 uses a real data set for two-year and nine-year caplets in the Japanese LIBOR market as of late May 1998. The figure shows both observed implied volatility curves and calibrated implied volatility curves derived by using the futures option formula.

3. The model can lead to analytical solutions to many option pricing problems, including

    - standard call and put options;
    - interest rate derivatives, such as caplets, caps and bond options;
    - path-dependent options, such as barrier options, lookback options and perpetual American options.

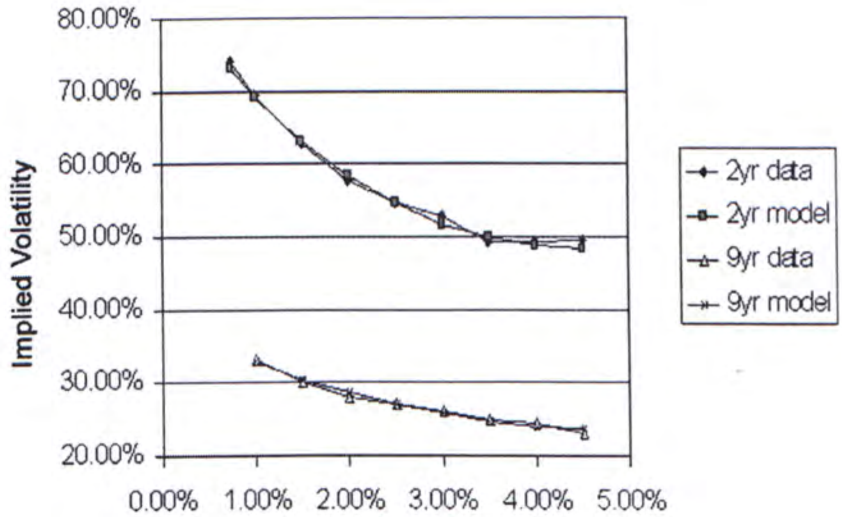

Figure 2.2: The Volatility Smile

4. The model has some (economical, physical, psychological, etc.) interpretation. One motivation comes from behavioral finance. Large stock market trends often begin and end with periods of frenzied buying (bubbles) or selling (crashes). Many observers cite these episodes as clear examples of herding behavior that is irrational and driven by emotion - greed in the bubbles, fear in the crashes. Individual investors join the crowd of others in a rush to get in or out of the market. Also, extensive empirical studies has suggested that markets tend to have both overreaction and underreaction to various good or bad news. Therefore, one may interpret the jump part of the model as the market response to outside news.

2.2.3 Preliminary Results

The infinitesimal generator of the jump diffusion process $X(t) = \log(S(t)/S(0))$ is given by

$$Lv(x) = \frac{1}{2}\sigma^2 v''(x) + (r - \frac{1}{2}\sigma^2 - \lambda\zeta)v'(x) + \lambda \int_{-\infty}^{+\infty} [v(x+y) - v(x)] f_Y(y) dy$$

for all twice continuously differentiable functions $v(x)$. In addition, suppose that $\theta \in (-\eta_2, \eta_1)$. The moment generating function of jump size Y is given by

$$E[e^{\theta Y}] = \frac{p\eta_1}{\eta_1 - \theta} + \frac{q\eta_2}{\eta_2 + \theta},$$

from which the moment generating function of X_t can be obtained as

$$\nu(\theta, t) := E[e^{\theta X_t}] = \exp\{G(\theta)t\},$$

where the function $G(\cdot)$ is defined as

$$G(x) := \frac{1}{2}\sigma^2 x^2 + (r - \frac{1}{2}\sigma^2 - \lambda\zeta)x + \lambda(\frac{p\eta_1}{\eta_1 - x} + \frac{q\eta_2}{\eta_2 + x} - 1).$$

Kou and Wang (2003) explores the property of function $G(x)$ and gives

Lemma 2.1 *The equation*

$$G(x) = m, \quad \text{for all } m > 0$$

has exactly four roots: $\beta_{1,m}$, $\beta_{2,m}$, $-\beta_{3,m}$, $-\beta_{4,m}$, *where*

$$0 < \beta_{1,m} < \eta_1 < \beta_{2,m} < \infty,$$

$$0 < \beta_{3,m} < \eta_2 < \beta_{4,m} < \infty.$$

Proof. Since $G(x)$ is a convex function on the interval $(-\eta_2, \eta_1)$ with $G(0) = \lambda(p + q - 1) = 0$, $G(\eta_1-) = +\infty$ and $G(-\eta_1+) = +\infty$, there is exactly one root $\beta_{1,m}$ for

$G(x) = m$ on the interval $(0, \eta_1)$, and another one on the interval $(-\eta_2, 0)$. Furthermore, since $G(\eta_1+) = -\infty$ and $G(+\infty) = +\infty$, there is at least one root on (η_1, ∞). Similarly, there is at least one root on $(-\infty, -\eta_2)$, as $G(-\eta_2-) = -\infty$ and $G(-\infty) = +\infty$. But the equation $G(x) = m$ is actually a polynomial equation with degree four; therefore, it can have at most four real roots. It follows that, on each interval, $(-\infty, -\eta_2)$ and (η_1, ∞), there is exactly one root.

2.2.4 Extant Results on Option Pricing under the Model

In this part, I will review some extant results on option pricing under the double exponential jump diffusion model.

For any given probability P, define

$$\Upsilon(\mu, \sigma, \lambda, p, \eta_1, \eta_2; a, T) \;\; := \;\; P(Z(T) \geq a),$$

where $Z(t) = \mu t + \sigma W(t) + \sum_{i=1}^{N(t)} Y_i$, Y has a double exponential distribution with density $f_Y(y) \sim p \cdot \eta_1 e^{-\eta_1 y} 1_{\{y \geq 0\}} + q \cdot \eta_2 e^{\eta_2 y} 1_{\{y<0\}}$, and $N(t)$ is a Poisson process with rate λ.

Kou (2002) shows that Υ can be derived as a sum of Hh functions, i.e.,

$$\begin{aligned}
\Upsilon(\mu, \sigma, \lambda, p, \eta_1, \eta_2; a, T) &= \pi_0 P(\mu T + \sigma\sqrt{T} Z \geq a) \\
&\quad + \sum_{n=1}^{\infty} \pi_n \sum_{k=1}^{n} P_{n,k} P(\mu T + \sigma\sqrt{T} Z + \sum_{j=1}^{k} \xi_j^+ \geq a) \\
&\quad + \sum_{n=1}^{\infty} \pi_n \sum_{k=1}^{n} Q_{n,k} P(\mu T + \sigma\sqrt{T} Z - \sum_{j=1}^{k} \xi_j^- \geq a),
\end{aligned}$$

where ξ_j^+ and ξ_j^- are i.i.d. exponential random variable with rates η_1 and η_2, respectively, and Z is a normal random variable with distribution $N(0, \sigma^2)$, and

$$\pi_n \;\; = \;\; e^{-\lambda T}(\lambda T)^n / n!,$$

$$
\begin{aligned}
P_{n,k} &= \sum_{i=k}^{n-1} \begin{pmatrix} n-k-1 \\ i-k \end{pmatrix} \begin{pmatrix} n \\ i \end{pmatrix} \left(\frac{\eta_1}{\eta_1+\eta_2}\right)^{i-k} \left(\frac{\eta_2}{\eta_1+\eta_2}\right)^{n-i} p^i q^{n-i}, \\
Q_{n,k} &= \sum_{i=k}^{n-1} \begin{pmatrix} n-k-1 \\ i-k \end{pmatrix} \begin{pmatrix} n \\ i \end{pmatrix} \left(\frac{\eta_1}{\eta_1+\eta_2}\right)^{n-i} \left(\frac{\eta_2}{\eta_1+\eta_2}\right)^{i-k} p^{n-i} q^{i}.
\end{aligned}
$$

(a) European call and put options

Suppose that the strike price is K and the maturity is T. In terms of Υ, the price of a European call option is given by

$$
\begin{aligned}
C(0,T) &= S(0)\Upsilon(r+\frac{1}{2}\sigma^2-\lambda\zeta,\sigma,\widetilde{\lambda},\widetilde{p},\widetilde{\eta}_1,\widetilde{\eta}_2;log(K/S(0)),T) \\
&\quad -Ke^{-rT}\Upsilon(r-\frac{1}{2}\sigma^2-\lambda\zeta,\sigma,\lambda,p,\eta_1,\eta_2;log(K/S(0)),T),
\end{aligned}
$$

where

$$
\widetilde{p} = \frac{p}{1+\zeta}\cdot\frac{\eta_1}{\eta_1-1}, \quad \widetilde{\eta}_1 = \eta_1 - 1,
$$
$$
\widetilde{\eta}_2 = \eta_2 + 1, \quad \widetilde{\lambda} = \lambda(\zeta+1), \quad \zeta = \frac{p\eta_1}{\eta_1-1} + \frac{q\eta_2}{\eta_2+1} - 1.
$$

(b) Lookback options

The price of a lookback put option can be formulated as

$$
LP(T) = E^*[e^{-rT}(\max\{M, \max_{0\le t\le T} S(t)\} - S(T))],
$$

where $M \ge S(0)$ is a fixed constant representing the prefixed maximum at time 0.

Hence, the Laplace transform of the lookback put is given by

$$
\int_0^\infty e^{-\alpha T} LP(T)dT = \frac{S(0)A_\alpha}{C_\alpha}\left(\frac{S(0)}{M}\right)^{\beta_{1,\alpha+r}-1} + \frac{S(0)B_\alpha}{C_\alpha}\left(\frac{S(0)}{M}\right)^{\beta_{2,\alpha+r}-1} + \frac{M}{\alpha+r} - \frac{S(0)}{\alpha},
$$

for all $\alpha > 0$, where

$$
A_\alpha = \frac{(\eta_1-\beta_{1,\alpha+r})\beta_{2,\alpha+r}}{\beta_{1,\alpha+r}-1}, \quad B_\alpha = \frac{(\beta_{2,\alpha+r}-\eta_1)\beta_{1,\alpha+r}}{\beta_{2,\alpha+r}-1},
$$
$$
C_\alpha = (\alpha+r)\eta_1(\beta_{2,\alpha+r}-\beta_{1,\alpha+r}).
$$

(c) Barrier options

Consider the up-and-in call (UIC) option, whose price is given by

$$UIC = E^*[e^{-rT}(S(T)-K)^+ 1_{\{\max_{0\le t\le T} S(t)\ge H\}}],$$

where $H > S(0)$ is the barrier level.

For any given probability P, define

$$\Psi(\mu,\sigma,\lambda,p,\eta_1,\eta_2;a,b,T) := P(Z(T)\ge a, \max_{0\le t\le T} Z(t)\ge b),$$

where $Z(t)=\mu t+\sigma W(t)+\sum_{i=1}^{N(t)} Y_i$, Y has a double exponential distribution with density $f_Y(y)\sim p\cdot\eta_1 e^{-\eta_1 y}1_{\{y\ge 0\}}+q\cdot\eta_2 e^{\eta_2 y}1_{\{y<0\}}$, and $N(t)$ is a Poisson process with rate λ. Like Υ, the Laplace transform of Ψ can also be derived as a sum of the Hh functions.

In terms of Ψ, the price of the UIC option is obtained as

$$\begin{aligned} UIC &= S(0)\Psi(r+\frac{1}{2}\sigma^2-\lambda\zeta,\sigma,\widetilde{\lambda},\widetilde{p},\widetilde{\eta}_1,\widetilde{\eta}_2;log(K/S(0)),log(H/S(0)),T) \\ &\quad -Ke^{-rT}\Psi(r-\frac{1}{2}\sigma^2-\lambda\zeta,\sigma,\lambda,p,\eta_1,\eta_2;log(K/S(0)),log(H/S(0)),T), \end{aligned}$$

where

$$\widetilde{p}=\frac{p}{1+\zeta}\cdot\frac{\eta_1}{\eta_1-1},\quad \widetilde{\eta}_1=\eta_1-1,$$
$$\widetilde{\eta}_2=\eta_2+1,\quad \widetilde{\lambda}=\lambda(\zeta+1),\quad \zeta=\frac{p\eta_1}{\eta_1-1}+\frac{q\eta_2}{\eta_2+1}-1.$$

(d) Perpetual American Options

Under the jump diffusion model, the price of an American put option is given by $\sup_\tau E^*[e^{-r\tau}(K-S(\tau))^+]$, where the supremum is taken over all stopping time τ taking values in $[0,\infty]$. The value of the perpetual American put option is given by $V(S(0))$, where the value function V is given by

$$V(v) = \begin{cases} K-v, & \text{if } v<v_0, \\ Av^{-\beta_{3,r}}+Bv^{-\beta_{4,r}}, & \text{if } v\ge v_0 \end{cases}$$

where

$$
\begin{aligned}
v_0 &= K\frac{\eta_2+1}{\eta_2}\cdot\frac{\beta_{3,r}}{1+\beta_{3,r}}\cdot\frac{\beta_{4,r}}{1+\beta_{4,r}}, \\
A &= v_0^{\beta_{3,r}}\frac{1+\beta_{4,r}}{\beta_{4,r}-\beta_{3,r}}[\frac{\beta_{4,r}}{1+\beta_{4,r}}K - v_0] > 0, \\
B &= v_0^{\beta_{4,r}}\frac{1+\beta_{3,r}}{\beta_{4,r}-\beta_{3,r}}[v_0 - \frac{\beta_{3,r}}{1+\beta_{3,r}}K] > 0.
\end{aligned}
$$

2.3 The Laplace Transform and Its Inversion

Laplace transforms have been widely used in valuing financial derivatives. For example, German and Yor (1993) uses Laplace transforms to price Asian options in the Black-Scholes setting; Laplace transforms for double-barrier and lookback options under the CEV model are given in Davydov and Linetsky (2001). In this thesis, the closed-form solutions of double barrier option prices are also derived in terms of Laplace transforms since the expressions of the transforms are easier to approach and have much simpler forms. Therefore, this section is mainly designed to review the basics of Laplace transform and its inversion algorithm.

2.3.1 The Laplace Transform

Suppose that f is a real- or complex-valued function of the variable $t > 0$ and s is a real of complex parameter. We define the Laplace transform of f as

$$
\begin{aligned}
\widehat{f}(s) = \mathcal{L}(f(t)) &= \int_0^\infty e^{-st}f(t)dt \\
&= \lim_{\tau\to\infty}\int_0^\tau e^{-st}f(t)dt \qquad (2.3)
\end{aligned}
$$

whenever the limit exists. When it does, the integral (2.3) is said to converge. If the limit does not exist, the integral is said to diverge and there is no Laplace transform defined for f. The notation $\mathcal{L}(f)$ is used to denote the Laplace transform of f, and the integral is the ordinary Riemann integral. The parameter s belongs to some domain on the real line or in the complex plane.

Example If $f(t) \equiv 1$ for $t \geq 0$, then

$$\begin{aligned} \mathcal{L}(f(t)) &= \int_0^\infty e^{-st} 1 dt \\ &= \lim_{\tau \to \infty} (\frac{e^{-s\tau}}{-s} + \frac{1}{s}) \\ &= \frac{1}{s} \end{aligned}$$

provided of course that $s > 0$ (if s is real). Thus we have

$$\mathcal{L}(1) = \frac{1}{s} \quad (s > 0).$$

If $s \leq 0$, then the integral would diverge and there would be no resulting Laplace transform. If we had taken s to be a complex variable, the same calculation, with $\mathrm{Re}(s) > 0$, would have given $\mathcal{L}(1) = 1/s$.

2.3.2 One-dimensional Euler Laplace Transform Inversion Algorithm

This method is proposed by Abate and Whitt (1992). The object is to calculate values of a real-valued function $f(t)$ of a positive real variable t for various t from the Laplace transform

$$\widehat{f}(s) = \int_0^\infty e^{-st} f(t) dt, \tag{2.4}$$

where s is a complex variable with nonnegative real part.

Letting the contour be any vertical line $s = a$ such that $\widehat{f}(s)$ has no singularities on or to the right of it, they obtain

$$f(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{st} \widehat{f}(s) ds$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(a+iu)t}\widehat{f}(a+iu)du \\
&= \frac{e^{at}}{2\pi}\int_{-\infty}^{\infty} (\cos ut + i\sin ut)\widehat{f}(a+iu)du \\
&= \frac{e^{at}}{2\pi}\int_{-\infty}^{\infty} [\mathrm{Re}(\widehat{f}(a+iu))\cos ut - \mathrm{Im}(\widehat{f}(a+iu))\sin ut]du \\
&= \frac{2e^{at}}{\pi}\int_{0}^{\infty} \mathrm{Re}(\widehat{f}(a+iu))\cos ut du, \qquad (2.5)
\end{aligned}
$$

where $i = \sqrt{-1}$ and $\mathrm{Re}(s)$ and $\mathrm{Im}(s)$ are the real and imaginary parts of s.

Evaluating the integral (2.5) by means of the trapezoidal rule and using a step size h give

$$
\begin{aligned}
f(t) \approx f_h(t) \equiv & \frac{he^{at}}{\pi}\mathrm{Re}(\widehat{f})(a) \\
& + \frac{2he^{at}}{\pi}\sum_{k=1}^{\infty}\mathrm{Re}(\widehat{f})(a+ikh)\cos(kht). \qquad (2.6)
\end{aligned}
$$

Letting $h = \pi/2t$ and $a = A/2t$, (2.6) can be written as nearly alternating series

$$
\begin{aligned}
f_h(t) = & \frac{e^{A/2}}{2t}\mathrm{Re}(\widehat{f})(\frac{A}{2t}) \\
& + \frac{e^{A/2}}{t}\sum_{k=1}^{\infty}(-1)^k\mathrm{Re}(\widehat{f})(\frac{A+2k\pi i}{2t}). \qquad (2.7)
\end{aligned}
$$

Abate and Whitt use the Poisson summation formula to identify the discretization error associated with (2.7). The essential idea is to replace the damped function $g(t) \equiv e^{-bt}f(t)$ for $b > 0$ by the periodic function

$$
g_p(t) = \sum_{k=-\infty}^{\infty} g(t + \frac{2\pi k}{h}) \qquad (2.8)
$$

of period $2\pi/h$. Representing the periodic function g_p by its complex Fourier series gives

$$
g_p(t) = \sum_{k=-\infty}^{\infty} c_k e^{ikht}, \qquad (2.9)
$$

where c_k is the kth Fourier coefficient of g_p, i.e.,

$$\begin{aligned} c_k &= \frac{h}{2\pi}\int_{-\pi/h}^{\pi/h} g_p(t)e^{-ikht}dt \\ &= \frac{h}{2\pi}\int_{-\pi/h}^{\pi/h} \sum_{k=-\infty}^{\infty} g(t+\frac{2\pi k}{h})e^{-ikht}dt \\ &= \frac{h}{2\pi}\int_{-\infty}^{\infty} g(t)e^{-ikht}dt \\ &= \frac{h}{2\pi}\int_{0}^{\infty} e^{-bt}f(t)e^{-ikht}dt \\ &= \frac{h}{2\pi}\widehat{f}(b+ikh). \end{aligned} \tag{2.10}$$

Combining (2.8)-(2.10), a version of Poisson summation formula is obtained

$$\begin{aligned} g_p(t) &= \sum_{k=-\infty}^{\infty} g(t+\frac{2\pi k}{h}) \\ &= \sum_{k=-\infty}^{\infty} f(t+\frac{2\pi k}{h})e^{-b(t+2\pi k/h)} \\ &= \frac{h}{2\pi}\sum_{k=-\infty}^{\infty} \widehat{f}(b+ikh)e^{ikht}. \end{aligned} \tag{2.11}$$

Let $h=\pi/t$ and $b=A/2t$ in (2.11) and obtain

$$\begin{aligned} f(t) &= \frac{e^{A/2}}{2t}\sum_{k=-\infty}^{\infty}(-1)^k\mathrm{Re}\widehat{f}(\frac{A+2k\pi i}{2t}) \\ &\quad - \sum_{k=-\infty,k\neq 0}^{\infty} e^{-kA}f((2k+1)t). \end{aligned} \tag{2.12}$$

Note that the first term on the right in (2.12) coincides with the trapezoidal-rule approximation in (2.7), so that the second term on the right in (2.12) gives the discretization error associated with the trapezoidal rule, i.e.,

$$e_d = \sum_{k=1}^{\infty} e^{-kA}f((2k+1)t). \tag{2.13}$$

If, as in probability applications $|f(t)|\leq 1$ for all t, then the error is bounded by

$$|e_d| \leq \frac{e^{-A}}{1-e^{-A}},$$

which is approximately equal to e^{-A} when e^{-A} is small. Hence, to have at most $10^{-\gamma}$ discretization error, let $A = \gamma \log 10$. ($A = 18.4$ is often used to achieve 10^{-8} discretization error.)

The remaining problem is to numerically calculate (2.7), which involves an infinite sum. Since the sum would be an alternating series if $\mathrm{Re}(f((A + 2k\pi i)/2t))$ would have constant sign for all k, Abate and Whitt suggest using Euler summation. Euler summation can be very simply described as the weighted average of the last m partial sums by a binomial probability distribution with parameters m and $p = 1/2$. In particular, let $s_n(t)$ by the approximation $f_h(t)$ in (2.7) with the infinite series truncated to n terms, i.e.,

$$s_n(t) = \frac{e^{A/2}}{2t}\mathrm{Re}(\widehat{f})(\frac{A}{2t}) + \frac{e^{A/2}}{t}\sum_{k=1}^{n}(-1)^k a_k(t),$$

where

$$a_k(t) = \mathrm{Re}(\widehat{f})(\frac{A + 2k\pi i}{2t}).$$

We apply Euler summation to m terms after an initial n, so that the Euler sum is

$$E(m, n, t) = \sum_{k=0}^{m} C_m^k 2^{-m} s_{n+k}(t).$$

Hence, $E(m, n, t)$ is the binomial average of the terms $s_n, s_{n+1}, \ldots, s_{n+m}$. (We typically use $m = 11$ and $n = 15$, increasing n as necessary.)

2.3.3 Two-dimensional Euler Laplace Transform Inversion Algorithm

In this section, we study the algorithm by Choudhury, Lucantoni and Whitt (1994) which numerically inverts a two-dimensional Laplace transform. Let $f(t_1, t_2)$ be a complex-valued function of nonnegative real variables t_1 and t_2, and let its two-dimensional Laplace transform be

$$\widehat{f}(s_1, s_2) = \int_0^\infty \int_0^\infty \exp(-(s_1 t_1 + s_2 t_2)) f(t_1, t_2) dt_1 dt_2,$$

which we assume is well defined.

The two-dimensional Poisson summation formula gives

$$\begin{aligned} & \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} g(t_1 + \frac{2\pi j}{h_1}, t_2 + \frac{2\pi k}{h_2}) \\ = & \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} f(t_1 + \frac{2\pi j}{h_1}, t_2 + \frac{2\pi k}{h_2}) \exp\{-[b_1(t_1 + 2\pi j/h_1) + b_2(t_2 + 2\pi k/h_2)]\} \\ = & \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \frac{h_1 h_2}{4\pi^2} \widehat{f}(b_1 - ijh_1, b_2 - ikh_2) \exp(-i(jh_1 t_1 + kh_2 t_2)). \qquad (2.14) \end{aligned}$$

Let $h_1 = \pi/(t_1 l_1)$ and $h_2 = \pi/(t_2 l_2)$, where $l_1, l_2 \geq 1$, then (2.14) becomes

$$\begin{aligned} & \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \exp(-[b_1(1 + 2jl_1)t_1 + b_2(1 + 2kl_2)t_2]) \\ & \times f((1 + 2jl_1)t_1, (1 + 2kl_2)t_2) \\ = & \frac{1}{4l_1 t_1 l_2 t_2} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \exp(-i(\frac{j\pi}{l_1} + \frac{k\pi}{l_2})) \\ & \times \widehat{f}(b_1 - \frac{ij\pi}{l_1 t_1}, b_2 - \frac{ik\pi}{l_2 t_2}). \end{aligned}$$

Furthermore, let $b_1 = A_1/(2t_1 l_1)$ and $b_2 = A_2/(2t_2 l_2)$ and get

$$\begin{aligned} & \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \exp(-(A_1 j + A_2 k)) f((1 + 2jl_1)t_1, (1 + 2kl_2)t_2) \\ = & \frac{\exp(A_1/(2l_1) + A_2/(2l_2))}{4l_1 t_1 l_2 t_2} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \exp(-i(\frac{j\pi}{l_1} + \frac{k\pi}{l_2})) \\ & \times \widehat{f}(\frac{A_1}{2l_1 t_1} - \frac{ij\pi}{l_1 t_1}, \frac{A_2}{2l_2 t_2} - \frac{ik\pi}{l_2 t_2}). \qquad (2.15) \end{aligned}$$

Note that we can rewrite (2.15) as $f(t_1, t_2) = f_h(t_1, t_2) - e_d$, where the value to be calculated is

$$\begin{aligned} f_h(t_1,t_2) &= \frac{\exp(A_1/(2l_1))}{2l_1t_1}\sum_{j=-\infty}^{\infty}\exp(-\frac{ij\pi}{l_1}) \\ &\times\{\frac{\exp(A_2/(2l_2))}{2l_2t_2}\sum_{k=-\infty}^{\infty}\exp(-\frac{ik\pi}{l_2}) \\ &\times[\widehat{f}(\frac{A_1}{2l_1t_1}-\frac{ij\pi}{l_1t_1},\frac{A_2}{2l_2t_2}-\frac{ik\pi}{l_2t_2})]\} \end{aligned} \tag{2.16}$$

and the error is

$$\begin{aligned} e_d &\equiv e_d(t_1,t_2,A_1,A_2,l_1,l_2) \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}{}_{not j=k=0}\exp(-(A_1j+A_2k))f((1+2jl_1)t_1,(1+2kl_2)t_2). \end{aligned}$$

We regard e_d as the error term, which will not be explicitly computed. If $|f(t_1,t_2)|\le C$ for some constant C and all t_1,t_2, then the error can be bounded as

$$|e_d|\le\frac{C(e^{-A_1}+e^{-A_2}-e^{-(A_1+A_2)})}{(1-e^{-A_1})(1-e^{-A_2})}\approx C(e^{-A_1}+e^{-A_2}).$$

This error may be reduced by increasing the parameters A_1 and A_2. For example, if $C=1$, then we can limit e_d to 10^{-8} by choosing $A_1=A_2=19.1$.

In order to be able to exploit the Euler summation technique for nearly alternating series, (2.16) is rewritten as

$$\begin{aligned} f_h(t_1,t_2) &= \frac{\exp(A_1/(2l_1))}{2l_1t_1}\sum_{j_1=1}^{l_1}\sum_{j=-\infty}^{\infty}(-1)^j\exp(-\frac{ij_1\pi}{l_1}) \\ &\times\{\frac{\exp(A_2/(2l_2))}{2l_2t_2}\sum_{k_1=1}^{l_2}\sum_{k=-\infty}^{\infty}(-1)^k\exp(-\frac{ik_1\pi}{l_2}) \\ &\times[\widehat{f}(\frac{A_1}{2l_1t_1}-\frac{ij_1\pi}{l_1t_1}-\frac{ij\pi}{t_1},\frac{A_2}{2l_2t_2}-\frac{ik_1\pi}{l_2t_2}-\frac{ik\pi}{t_2})]\}. \end{aligned} \tag{2.17}$$

If f is real valued, then the computation of (2.17) can be reduced by a factor of 2 by noting that $\widehat{f}(\overline{s_1},\overline{s_2})=\overline{\widehat{f}(s_1,s_2)}$, where s is the complex conjugate of s. Then (2.17) can be expressed as

$$
\begin{aligned}
f_h(t_1,t_2) &= \frac{\exp(A_1/(2l_1)+A_2/(2l_2))}{4t_1l_1t_2l_2} \\
&\times(\widehat{f}(\frac{A_1}{2t_1l_1},\frac{A_2}{2t_2l_2})+2\sum_{k_1=1}^{l_2}\sum_{k=0}^{\infty}(-1)^k\mathrm{Re}[\exp(\frac{-ik_1\pi}{l_1})\widehat{f}(\frac{A_1}{2t_1l_1},\frac{A_2}{2t_2l_2}-\frac{ik_1\pi}{t_2l_2}-\frac{ik\pi}{t_2})]) \\
&+2\sum_{j_1=1}^{l_1}\sum_{j=0}^{\infty}(-1)^j\mathrm{Re}[\sum_{k_1=1}^{l_2}\sum_{k=0}^{\infty}(-1)^k\exp(-(\frac{ij_1\pi}{l_1}+\frac{ik_1\pi}{l_2})) \\
&\times\widehat{f}(\frac{A_1}{2t_1l_1}-\frac{ij_1\pi}{t_1l_1}-\frac{ij\pi}{t_1},\frac{A_2}{2t_2l_2}-\frac{ik_1\pi}{t_2l_2}-\frac{ik\pi}{t_2})]+2\sum_{j_1=1}^{l_1}\sum_{j=0}^{\infty}(-1)^j\mathrm{Re}[\exp(-\frac{ij_1\pi}{l_1}) \\
&\times\widehat{f}(\frac{A_1}{2t_1l_1}-\frac{ij_1\pi}{t_1l_1}-\frac{ij\pi}{t_1},\frac{A_2}{2t_2l_2})+\sum_{k_1=1}^{l_2}\sum_{k=0}^{\infty}(-1)^k\exp(-(\frac{ij_1\pi}{l_1}-\frac{ik_1\pi}{l_2})) \\
&\times\widehat{f}(\frac{A_1}{2t_1l_1}-\frac{ij_1\pi}{t_1l_1}-\frac{ij\pi}{t_1},\frac{A_2}{2t_2l_2}+\frac{ik_1\pi}{t_2l_2}+\frac{ik\pi}{t_2})].
\end{aligned}
$$

Numerical experience indicates that with $l_1=l_2=1$ the result can usually achieve an overall accuracy of 5 or 6 digits, and with $l_1=l_2=2$, it can usually achieve an overall accuracy of 10 or more digits. Assuming that $l_1=l_2=1$, we can rewrite f_h as

$$
\begin{aligned}
f_h(t_1,t_2) &= \frac{\exp(A_1/2+A_2/2)}{4t_1t_2} \\
&\times(\widehat{f}(\frac{A_1}{2t_1},\frac{A_2}{2t_2})+2\sum_{k=0}^{\infty}(-1)^k\mathrm{Re}[\exp(-i\pi)\widehat{f}(\frac{A_1}{2t_1},\frac{A_2}{2t_2}-\frac{i\pi}{t_2}-\frac{ik\pi}{t_2})]) \\
&+2\sum_{j=0}^{\infty}(-1)^j\mathrm{Re}[\sum_{k=0}^{\infty}(-1)^k\exp(-2\pi i) \\
&\times\widehat{f}(\frac{A_1}{2t_1}-\frac{i\pi}{t_1}-\frac{ij\pi}{t_1},\frac{A_2}{2t_2}-\frac{i\pi}{t_2}-\frac{ik\pi}{t_2})]+2\sum_{j=0}^{\infty}(-1)^j\mathrm{Re}[\exp(-i\pi) \\
&\times\widehat{f}(\frac{A_1}{2t_1}-\frac{i\pi}{t_1}-\frac{ij\pi}{t_1},\frac{A_2}{2t_2})+\sum_{k=0}^{\infty}(-1)^k\times\widehat{f}(\frac{A_1}{2t_1}-\frac{i\pi}{t_1}-\frac{ij\pi}{t_1},\frac{A_2}{2t_2}+\frac{i\pi}{t_2}+\frac{ik\pi}{t_2})] \\
&= \frac{\exp(A_1/2+A_2/2)}{4t_1t_2} \\
&\times(\widehat{f}(\frac{A_1}{2t_1},\frac{A_2}{2t_2})+2\sum_{k=1}^{\infty}(-1)^{k-1}\mathrm{Re}[\exp(-i\pi)\widehat{f}(\frac{A_1}{2t_1},\frac{A_2}{2t_2}-\frac{ik\pi}{t_2})])
\end{aligned}
$$

$$+2\sum_{j=1}^{\infty}(-1)^{j-1}\mathrm{Re}[\sum_{k=1}^{\infty}(-1)^{k-1}\exp(-2\pi)$$
$$\times\widehat{f}(\frac{A_1}{2t_1}-\frac{ij\pi}{t_1},\frac{A_2}{2t_2}-\frac{ik\pi}{t_2})]+2\sum_{j=1}^{\infty}(-1)^{j-1}\mathrm{Re}[\exp(-i\pi)$$
$$\times\widehat{f}(\frac{A_1}{2t_1}-\frac{ij\pi}{t_1},\frac{A_2}{2t_2})+\sum_{k=1}^{\infty}(-1)^{k-1}\times\widehat{f}(\frac{A_1}{2t_1}-\frac{ij\pi}{t_1},\frac{A_2}{2t_2}+\frac{ik\pi}{t_2})]. \qquad (2.18)$$

Note that (2.18) contains infinite sums of the form $S=\sum_{k=0}^{\infty}(-1)^k a_k$, where a_k is real or complex. Hence, the Euler sum with parameters n and m is given by

$$E(m,n) \;=\; \sum_{k=0}^{m} C_m^k 2^{-m} S_{n+k}.$$

Choudhury, Lucantoni and Whitt shows that $E(m,n)$ computes S with an error of the order of 10^{-13} or less with the choice $n=38$ and $m=11$, that is, requiring the computation of only 50 terms.

2.4 Monte Carlo Simulation for Double Exponential Jump Diffusion

Assume that the process has the form

$$X(t) \;=\; \mu t+\sigma W(t)+\sum_{i=1}^{N(t)}Y_i, \quad X(0)=0.$$

We consider simulating the return process at a fixed set of dates $0=t_0<t_1<...<t_n$ by the recursion

$$X(t_{i+1}) \;=\; X(t_i)+\mu(t_{i+1}-t_i)+\sigma[W(t_{i+1})-W(t_i)]+\sum_{j=N(t_i)+1}^{N(t_{i+1})}Y_j \qquad (2.19)$$

A general method for simulating (2.19) from t_i to t_{i+1} consists of the following steps:

1. generate $Z\sim N(0,1)$

2. generate $N \sim \text{Poisson}(\lambda(t_{i+1} - t_i))$; if $N = 0$, set $M = 0$ and go to step 4

3. generate $Y_1, ..., Y_N$ from their common distribution and set $M = Y_1 + ... + Y_N$

4. set

$$X(t_{i+1}) = X(t_i) + \mu(t_{i+1} - t_i) + \sigma\sqrt{t_{i+1} - t_i}Z + M.$$

Chapter 3

Pricing Double Barrier Option via Laplace Transform

In this chapter, I begin to study the first passage time of the double barrier pricing problem for the double exponential jump diffusion model. Following a similar method as in Kou and Wang (2003), the Laplace transform of the first passage time is derived. In the last section of this chapter, I present the closed-form solution of the two-dimensional Laplace transform of the double barrier option price.

3.1 Double Barrier Option and the First Passage Time

Suppose that the initial asset price is S and the lower and upper barrier levels are L and U, $L < S < U$. Double barrier (double knock-out) options are canceled when the underlying asset first reaches either the upper or lower barrier.

We denote t the running time variable and assume that all options are written at time $t = 0$ and expire at time $t = T > 0$. Define the first stopping time of the lower barrier

$$\tau_L \; := \; \inf\{t \geq 0; \quad S_t \leq L\},$$

the first stopping time of the upper barrier

$$\tau_U \; := \; \inf\{t \geq 0; \quad S_t \geq U\},$$

and the first exit time from an interval between the two barriers

$$\tau_{(L,U)} \; := \; \inf\{t \geq 0; \quad S_t \leq L \quad or \quad S_t \geq U\}.$$

3.2 Preliminary Results

Under the risk-neutral measure, the double exponential jump diffusion model gives

$$\frac{dS(t)}{S(t-)} \; = \; (r - \lambda\zeta)dt + \sigma dW(t) + d(\sum_{i=1}^{N(t)}(V_i - 1)), \tag{3.1}$$

with the return process $X(t) = \log(S(t)/S(0))$ given by

$$X(t) \; = \; (r - \frac{1}{2}\sigma^2 - \lambda\zeta)t + \sigma W(t) + \sum_{i=1}^{N(t)} Y_i, \quad X(0) = 0. \tag{3.2}$$

The following result shows that the memoryless property of the random walk of exponential random variables leads to the conditional memoryless property of the jump diffusion process.

Proposition 3.1 *For any $x > 0$,*

$$\begin{aligned} P(\tau_{(L,U)} \le t, X_{\tau_{(L,U)}} - H \ge x) &= e^{-\eta_1 x} P(\tau_{(L,U)} \le t, X_{\tau_{(L,U)}} - H > 0), \quad (3.3) \\ P(X_{\tau_{(L,U)}} - H \ge x | X_{\tau_{(L,U)}} - H > 0) &= e^{-\eta_1 x}. \quad (3.4) \end{aligned}$$

Furthermore, conditional on $X_{\tau_{(L,U)}} - H > 0$, the stopping time $\tau_{(L,U)}$ and the overshoot $X_{\tau_{(L,U)}} - H$ are independent; more precisely, for any $x > 0$,

$$\begin{aligned} & P(\tau_{(L,U)} \le t, X_{\tau_{(L,U)}} - H \ge x | X_{\tau_{(L,U)}} - H > 0) \\ = \; & P(\tau_{(L,U)} \le t | X_{\tau_{(L,U)}} - H > 0) P(X_{\tau_{(L,U)}} - H \ge x | X_{\tau_{(L,U)}} - H > 0). \quad (3.5) \end{aligned}$$

Proof. We only need to show that (3.3) holds. The equality (3.4) follows readily since on the set $\{X_{\tau_{(L,U)}} > H\}$, the hitting time $\tau_{(L,U)}$ is finite by definition and letting $t \to \infty$, we have

$$\begin{aligned} & P(X_{\tau_{(L,U)}} - H \ge x | X_{\tau_{(L,U)}} - H > 0) \\ = \; & \frac{P(X_{\tau_{(L,U)}} - H \ge x, X_{\tau_{(L,U)}} - H > 0)}{P(X_{\tau_{(L,U)}} - H > 0)} \\ = \; & \frac{P(X_{\tau_{(L,U)}} - H \ge x)}{P(X_{\tau_{(L,U)}} - H > 0)} \\ = \; & \frac{\lim_{t\to\infty} P(\tau_{(L,U)} \le t, X_{\tau_{(L,U)}} - H \ge x)}{\lim_{t\to\infty} P(\tau_{(L,U)} \le t, X_{\tau_{(L,U)}} - H > 0)} \\ = \; & e^{-\eta_1 x}. \end{aligned}$$

And (3.5) also holds since

$$P(\tau_{(L,U)} \le t, X_{\tau_{(L,U)}} - H \ge x | X_{\tau_{(L,U)}} - H > 0)$$

$$
\begin{aligned}
&= \frac{P(\tau_{(L,U)} \leq t, X_{\tau_{(L,U)}} - H \geq x)}{P(X_{\tau_{(L,U)}} - H > 0)} \\
&= e^{-\eta_1 x} \frac{P(\tau_{(L,U)} \leq t, X_{\tau_{(L,U)}} - H > 0)}{P(X_{\tau_{(L,U)}} - H > 0)} \\
&= P(\tau_{(L,U)} \leq t | X_{\tau_{(L,U)}} - H > 0) P(X_{\tau_{(L,U)}} - H \geq x | X_{\tau_{(L,U)}} - H > 0).
\end{aligned}
$$

Denote by $T_1, T_2, ...$ the arrival times of the Poisson process N. It follows that

$$
P(\tau_{(L,U)} \leq t, X_{\tau_{(L,U)}} - H \geq x) = \sum_{n=1}^{\infty} P(T_n = \tau_{(L,U)}, X_{T_n} - H \geq x) =: \sum_{n=1}^{\infty} P_n,
$$

as the overshoot inside the probability can only occur during the arrival times of the Poisson process because $x > 0$. However, with $X_0(t) = \sigma W_t + \mu t$, we have

$$
\begin{aligned}
P_n &= P(\max_{0 \leq s < T_n} X_s < H, \min_{0 \leq s < T_n} X_s > h, X_{T_n} \geq H + x, T_n \leq t) \\
&= E[P(X_{T_n} \geq H + x | F_{T_n -}, T_n) 1_{\{\max_{0 \leq s < T_n} X_s < H, \min_{0 \leq s < T_n} X_s > h, X_{T_n} \geq H + x, T_n \leq t\}}] \\
&= E[p \exp\{-\eta_1 (H - X_0(T_n) - Y_1 - ... - Y_{n-1})\} 1_{\{\max_{0 \leq s < T_n} X_s < H, \min_{0 \leq s < T_n} X_s > h, X_{T_n} \geq H + x, T_n \leq t\}}] \\
&= e^{-\eta_1 x} E[P(X_{T_n} > H | F_{T_n -}, T_n) 1_{\{\max_{0 \leq s < T_n} X_s < H, \min_{0 \leq s < T_n} X_s > h, X_{T_n} \geq H + x, T_n \leq t\}}] \\
&= e^{-\eta_1 x} P(\max_{0 \leq s < T_n} X_s < H, \min_{0 \leq s < T_n} X_s > h, X_{T_n} > H, T_n \leq t) \\
&= e^{-\eta_1 x} P(\tau_{(L,U)} \leq t, X_{\tau_{(L,U)}} - H > 0).
\end{aligned}
$$

It follows that

$$
P(\tau_{(L,U)} \leq t, X_{\tau_{(L,U)}} - H \geq x) = e^{-\eta_1 x} P(\tau_{(L,U)} \leq t, X_{\tau_{(L,U)}} - H > 0),
$$

which completes the proof.

Corollary 3.2.1 *For any $x > 0$,*

$$
\begin{aligned}
P(\tau_{(L,U)} \leq t, X_{\tau_{(L,U)}} - h \leq x) &= e^{\eta_2 x} P(\tau_{(L,U)} \leq t, X_{\tau_{(L,U)}} - h < 0), \\
P(X_{\tau_{(L,U)}} - h \leq x | X_{\tau_{(L,U)}} - h > 0) &= e^{\eta_2 x}.
\end{aligned}
$$

Furthermore, conditional on $X_{\tau_{(L,U)}} - h < 0$, the stopping time $\tau_{(L,U)}$ and the overshoot $X_{\tau_{(L,U)}} - h$ are independent; more precisely, for any $x > 0$,

$$\begin{aligned} & P(\tau_{(L,U)} \le t, X_{\tau_{(L,U)}} - h \le x | X_{\tau_{(L,U)}} - h < 0) \\ = \; & P(\tau_{(L,U)} \le t | X_{\tau_{(L,U)}} - h < 0) P(X_{\tau_{(L,U)}} - h \le x | X_{\tau_{(L,U)}} - h < 0). \end{aligned}$$

3.3 Laplace Transform of the First Passage Time

The infinitesimal generator of the jump diffusion process $X(t) = \log(S(t)/S(0))$ is given by

$$Lv(x) = \frac{1}{2}\sigma^2 v''(x) + (r - \frac{1}{2}\sigma^2 - \lambda\zeta)v'(x) + \lambda \int_{-\infty}^{+\infty} [v(x+y) - v(x)] f_Y(y)dy$$

for all twice continuously differentiable functions $v(x)$. In addition, suppose that $\theta \in (-\eta_2, \eta_1)$. The moment generating function of jump size Y is given by

$$E[e^{\theta Y}] = \frac{p\eta_1}{\eta_1 - \theta} + \frac{q\eta_2}{\eta_2 + \theta},$$

from which the moment generating function of X_t can be obtained as

$$\nu(\theta, t) := E[e^{\theta X_t}] = \exp\{G(\theta)t\},$$

where the function $G(\cdot)$ is defined as

$$G(x) := \frac{1}{2}\sigma^2 x^2 + (r - \frac{1}{2}\sigma^2 - \lambda\zeta)x + \lambda(\frac{p\eta_1}{\eta_1 - x} + \frac{q\eta_2}{\eta_2 + x} - 1).$$

As Lemma 2.1 shows, the equation

$$G(x) = m, \quad \text{for all } m > 0$$

has exactly four roots: $\beta_{1,m}, \beta_{2,m}, -\beta_{3,m}, -\beta_{4,m}$, where

$$0 < \beta_{1,m} < \eta_1 < \beta_{2,m} < \infty,$$

$$0 < \beta_{3,m} < \eta_2 < \beta_{4,m} < \infty.$$

From now on, we suppose that the initial asset price is S which satisfies $L < S < U$, and let $H = log(\frac{U}{S})$ and $h = log(\frac{L}{S})$.

Proposition 3.2 *Let $\beta_{1,m}$, $\beta_{2,m}$, $-\beta_{3,m}$, $-\beta_{4,m}$ be the only four roots of the equation*

$$G(\beta) \;=\; m,$$

where $0 < \beta_{1,m} < \eta_1 < \beta_{2,m} < \infty$ and $0 < \beta_{3,m} < \eta_2 < \beta_{4,m} < \infty$. Let

$$\omega \;=\; \left[\; (\tfrac{S}{U})^{\beta_{1,m}},\;\; (\tfrac{S}{U})^{\beta_{2,m}},\;\; (\tfrac{L}{S})^{\beta_{3,m}},\;\; (\tfrac{L}{S})^{\beta_{4,m}} \;\right]$$

and

$$M \;=\; \begin{bmatrix} 1 & 1 & (\frac{L}{U})^{\beta_{3,m}} & (\frac{L}{U})^{\beta_{4,m}} \\ (\frac{L}{U})^{\beta_{1,m}} & (\frac{L}{U})^{\beta_{2,m}} & 1 & 1 \\ \frac{\eta_1}{\eta_1-\beta_{1,m}} & \frac{\eta_1}{\eta_1-\beta_{2,m}} & \frac{\eta_1}{\eta_1+\beta_{3,m}}(\frac{L}{U})^{\beta_{3,m}} & \frac{\eta_1}{\eta_1+\beta_{4,m}}(\frac{L}{U})^{\beta_{4,m}} \\ \frac{\eta_2}{\eta_2+\beta_{1,m}}(\frac{L}{U})^{\beta_{1,m}} & \frac{\eta_2}{\eta_2+\beta_{2,m}}(\frac{L}{U})^{\beta_{2,m}} & \frac{\eta_2}{\eta_2-\beta_{3,m}} & \frac{\eta_2}{\eta_2-\beta_{4,m}} \end{bmatrix}$$

Then we have the following results concerning the Laplace transforms of τ_L, τ_U and $\tau_{(L,U)}$:

$$E[e^{-m\tau_U} 1_{\{\tau_U<\tau_L\}}] \;=\; \left[\; A_1,\;\; B_1,\;\; C_1,\;\; D_1 \;\right]\omega^T, \tag{3.6}$$

$$E[e^{-m\tau_U} 1_{\{X_{\tau_{(L,U)}}-H>z\}}] \;=\; \left[\; A_{11},\;\; B_{11},\;\; C_{11},\;\; D_{11} \;\right]\omega^T \;\; \textit{for all } z \geq 0, \tag{3.7}$$

$$E[e^{-m\tau_U} 1_{\{X_{\tau_{(L,U)}}=H\}}] \;=\; \left[\; A_{12},\;\; B_{12},\;\; C_{12},\;\; D_{12} \;\right]\omega^T, \tag{3.8}$$

where A_1, B_1, C_1, D_1 satisfy

$$M\left[\; A_1,\;\; B_1,\;\; C_1,\;\; D_1 \;\right]^T \;=\; \left[\; 1,\;\; 0,\;\; 1,\;\; 0 \;\right]^T, \tag{3.9}$$

and A_{11}, B_{11}, C_{11}, D_{11} satisfy

$$M\left[\; A_{11},\;\; B_{11},\;\; C_{11},\;\; D_{11} \;\right]^T \;=\; \left[\; 0,\;\; 0,\;\; e^{-\eta_1 z},\;\; 0 \;\right]^T, \tag{3.10}$$

and A_{12}, B_{12}, C_{12}, D_{12} satisfy

$$A_1 = A_{11} + A_{12},\;\; B_1 = B_{11} + B_{12},\;\; C_1 = C_{11} + C_{12}, D_1 = D_{11} + D_{12}.$$

Proof. For simplicity, we write $\beta_{i,m}$ as β_i, $i = 1, 2, 3, 4$.

We first prove (3.6). Define the function v to be

$$v(x) := \begin{cases} A_1 e^{-\beta_1(H-x)} + B_1 e^{-\beta_2(H-x)} + C_1 e^{\beta_3(h-x)} + D_1 e^{\beta_4(h-x)}, & h < x < H, \\ 1, & x \geq H, \\ 0, & x \leq h, \end{cases}$$

where A_1, B_1, C_1 and D_1 are the solutions of (3.9). Clearly, $v(x)$ is bounded for all $x \in (-\infty, +\infty)$. Suppose that $|v(x)| \leq k < \infty$. Furthermore, the function v is continuous.

Substituting the form of v and doing the integration in three regions, $\int_{-\infty}^{+\infty} = \int_{-\infty}^{h-x} + \int_{h-x}^{H-x} + \int_{H-x}^{+\infty}$, we have that, for all $h < x < H$,

$$\begin{aligned} -mv + Lv \;=\; & A_1 e^{-\beta_1(H-x)} f(\beta_1) + B_1 e^{-\beta_2(H-x)} f(\beta_2) + C_1 e^{\beta_3(h-x)} f(\beta_3) + D_1 e^{\beta_4(h-x)} f(\beta_4) - \\ & \lambda p e^{-\eta_1(H-x)} [A_1 \frac{\eta_1}{\eta_1 - \beta_1} + B_1 \frac{\eta_1}{\eta_1 - \beta_2} + C_1 \frac{\eta_1}{\eta_1 + \beta_3} (\frac{L}{U})^{\beta_3} + D_1 \frac{\eta_1}{\eta_1 + \beta_4} (\frac{L}{U})^{\beta_4} - 1] - \\ & \lambda q e^{\eta_2(h-x)} [A_1 \frac{\eta_2}{\eta_2 + \beta_1} (\frac{L}{U})^{\beta_1} + B_1 \frac{\eta_2}{\eta_2 + \beta_2} (\frac{L}{U})^{\beta_2} + C_1 \frac{\eta_2}{\eta_2 - \beta_3} + D_1 \frac{\eta_2}{\eta_2 - \beta_4}], \end{aligned}$$

where $f(\beta) := G(\beta) - m$. From (3.9) we know that

$$\begin{aligned} A_1 \frac{\eta_1}{\eta_1 - \beta_1} + B_1 \frac{\eta_1}{\eta_1 - \beta_2} + C_1 \frac{\eta_1}{\eta_1 + \beta_3} (\frac{L}{U})^{\beta_3} + D_1 \frac{\eta_1}{\eta_1 + \beta_4} (\frac{L}{U})^{\beta_4} &= 1, \\ A_1 \frac{\eta_2}{\eta_2 + \beta_1} (\frac{L}{U})^{\beta_1} + B_1 \frac{\eta_2}{\eta_2 + \beta_2} (\frac{L}{U})^{\beta_2} + C_1 \frac{\eta_2}{\eta_2 - \beta_3} + D_1 \frac{\eta_2}{\eta_2 - \beta_4} &= 0. \end{aligned}$$

We also know that

$$f(\beta_1) = f(\beta_2) = f(-\beta_3) = f(-\beta_4) = 0.$$

Accordingly, we have

$$-mv(x) + Lv(x) \;=\; 0 \quad \textit{for all } h < x < H. \tag{3.11}$$

Because the function $v(x)$ is continuous, but not C^1 neither at $x = H$ nor $x = h$, we cannot apply Itô's formula directly to the process $\{e^{-mt} v(X_t); t \geq 0\}$. However, it is not difficult to see that there exists a sequence of functions $\{v_n(x); n = 1, 2, ...\}$ such that: (i) $v_n(x)$ is smooth everywhere, and in particular it belongs to C^2; (ii) $v_n(x) = v(x)$ for all $h \le x \le H$; (iii) $v_n(x) = 1 = v(x)$ for all $x \ge H + 1/n$; (iv) $v_n(x) = 0 = v(x)$ for all $x \le h - 1/n$; (v) $|v_n(x)| \le 2k$ for all x and n. Clearly, $v_n(x) \to v(x)$ for all x.

Hence, for $h < x < H$,

$$\begin{aligned}
Lv_n(x) &= \frac{1}{2}\sigma^2 v_n''(x) + (r - \frac{1}{2}\sigma^2 - \lambda\zeta)v_n'(x) + \lambda\int_{-\infty}^{\infty}[v_n(x+y) - v_n(x)]f_Y(y)dy \\
&= \frac{1}{2}\sigma^2 v_n''(x) + (r - \frac{1}{2}\sigma^2 - \lambda\zeta)v_n'(x) - \lambda v_n(x) + \lambda\int_{-\infty}^{\infty} v_n(x+y)f_Y(y)dy \\
&= \frac{1}{2}\sigma^2 v''(x) + (r - \frac{1}{2}\sigma^2 - \lambda\zeta)v'(x) - \lambda v(x) + \lambda\int_{h-x}^{H-x} v(x+y)f_Y(y)dy + \\
&\quad \lambda\int_{H-x}^{H-x+1/n} v_n(x+y)f_Y(y)dy + \lambda\int_{H-x+1/n}^{\infty} v(x+y)f_Y(y)dy + \\
&\quad \lambda\int_{h-x-1/n}^{h-1/n} v_n(x+y)f_Y(y)dy + \lambda\int_{-\infty}^{h-x} v(x+y)f_Y(y)dy \\
&= mv(x) + \lambda\int_{H-x}^{H-x+1/n}[v_n(x+y) - v(x+y)]f_Y(y)dy + \\
&\quad \lambda\int_{h-x-1/n}^{h-1/n}[v_n(x+y) - v(x+y)]f_Y(y)dy.
\end{aligned}$$

Since $|v_n - v| \le 4k$ by construction, it follows that, for all $h < x < H$,

$$\begin{aligned}
|-mv_n(x) + Lv_n(x)| &\le \lambda p\int_{H-x}^{H-x+1/n}|v_n(x+y) - v(x+y)|\eta_1 dy \\
&\quad +\lambda q\int_{h-x-1/n}^{h-x}|v_n(x+y) - v(x+y)|\eta_2 dy \\
&\le \frac{4\lambda k(p\eta_1 + q\eta_2)}{n} \to 0, \ as\ n \to \infty \qquad (3.12)
\end{aligned}$$

Applying the Itô's formula for jump processes to the process $\{e^{-mt}v_n(X_t); t \ge 0\}$, we obtain that the process

$$M_t^{(n)} := e^{-m(t\wedge\tau_{(L,U)})}v_n(X_{t\wedge\tau_{(L,U)}}) - \int_0^{t\wedge\tau_{(L,U)}} e^{-ms}(-mv_n(X_s) + Lv_n(X_s))ds, \quad t \ge 0,$$

is a local martingale starting from $M_0^{(n)} = v_n(0) = v(0)$. However, $|M_t^{(n)}| \leq 2k + \frac{4\lambda k(p\eta_1+q\eta_2)}{n}t$ for all $t \geq 0$, which indicates that $\{M_t^{(n)}; t \geq 0\}$ is actually a martingale. In particular, $EM_t^{(n)} = v(0)$ for all $t \geq 0$.

Letting $n \to \infty$, it follows from the dominated convergence theorem that

$$\lim_{n\to\infty} E[e^{-m(t\wedge\tau_{(L,U)})}v_n(X_{t\wedge\tau_{(L,U)}})] \;=\; E[e^{-m(t\wedge\tau_{(L,U)})}v(X_{t\wedge\tau_{(L,U)}})]$$

and, according to (3.12),

$$\begin{aligned}
& \lim_{n\to\infty} E[\int_0^{t\wedge\tau_{(L,U)}} e^{-ms}(-mv_n(X_s) + Lv_n(X_s))ds] \\
= & \lim_{n\to\infty} E[\int_0^{t\wedge\tau_{(L,U)}-} e^{-ms}(-mv_n(X_s) + Lv_n(X_s))ds] \\
= & \; 0.
\end{aligned}$$

Therefore, for any $t \geq 0$,

$$\begin{aligned}
v(0) &= E[e^{-m(t\wedge\tau_{(L,U)})}v(X_{t\wedge\tau_{(L,U)}})] \\
&= E[e^{-m(t\wedge\tau_{(L,U)})}v(X_{t\wedge\tau_{(L,U)}})1_{\{\tau_{(L,U)}<\infty\}}] + E[e^{-mt}v(X_t)1_{\{\tau_{(L,U)}=\infty\}}].
\end{aligned}$$

Now letting $t \to \infty$, we have,

$$\begin{aligned}
v(0) &= E[e^{-m\tau_{(L,U)}}v(X_{\tau_{(L,U)}})1_{\{\tau_{(L,U)}<\infty\}}] \\
&= E[e^{-m\tau_{(L,U)}}1_{\{X_{\tau_{(L,U)}}\geq H\}}1_{\{\tau_{(L,U)}<\infty\}}] \\
&= E[e^{-m\tau_{(L,U)}}1_{\{X_{\tau_{(L,U)}}\geq H\}}] \\
&= E[e^{-m\tau_{(L,U)}}1_{\{\tau_U<\tau_L\}}] \\
&= E[e^{-m\tau_U}1_{\{\tau_U<\tau_L\}}],
\end{aligned}$$

as $v(X_{\tau_{(L,U)}}) = 1_{\{\tau_U<\tau_L\}}$ on the set $\{\tau_{(L,U)} < \infty\}$, from which the result follows.

Now we focus on the proof for (3.7). (3.8) follows immediately since letting $z = 0$,

$$E[e^{-m\tau_U}1_{\{X_{\tau_{(L,U)}}=H\}}] \;=\; E[e^{-m\tau_U}1_{\{\tau_U<\tau_L\}}] - E[e^{-m\tau_U}1_{\{X_{\tau_{(L,U)}}-H>0\}}].$$

It suffices to consider the case where $z > 0$, as the case for $y = 0$ follows by letting $y \downarrow 0$. Define the function v to be

$$v(x) := \begin{cases} 1, & x > H + z, \\ 0, & H < x \leq H + z, \\ A_{11}e^{-\beta_1(H-x)} + B_{11}e^{-\beta_2(H-x)} + C_{11}e^{\beta_3(h-x)} + D_{11}e^{\beta_4(h-x)}, & h \leq x \leq H, \\ 0, & x < h, \end{cases}$$

where A_{11}, B_{11}, C_{11} and D_{11} are the solutions of the equations (3.10). Substitute to obtain that, for all $h \leq x \leq H$,

$$\begin{aligned} -mv + Lv &= A_{11}e^{-\beta_1(H-x)}f(\beta_1) + B_{11}e^{-\beta_2(H-x)}f(\beta_2) + C_{11}e^{\beta_3(h-x)}f(\beta_3) + D_{11}e^{\beta_4(h-x)}f(\beta_4) - \\ &\quad \lambda p e^{-\eta_1(H-x)}[A_{11}\frac{\eta_1}{\eta_1 - \beta_1} + B_{11}\frac{\eta_1}{\eta_1 - \beta_2} + \\ &\quad C_{11}\frac{\eta_1}{\eta_1 + \beta_3}(\frac{L}{U})^{\beta_3} + D_{11}\frac{\eta_1}{\eta_1 + \beta_4}(\frac{L}{U})^{\beta_4} - e^{-\eta_1 z}] - \\ &\quad \lambda q e^{\eta_2(h-x)}[A_{11}\frac{\eta_2}{\eta_2 + \beta_1}(\frac{L}{U})^{\beta_1} + B_{11}\frac{\eta_2}{\eta_2 + \beta_2}(\frac{L}{U})^{\beta_2} + C_{11}\frac{\eta_2}{\eta_2 - \beta_3} + D_{11}\frac{\eta_2}{\eta_2 - \beta_4}]. \end{aligned}$$

From (3.10) we know that

$$\begin{aligned} A_{11}\frac{\eta_1}{\eta_1 - \beta_1} + B_{11}\frac{\eta_1}{\eta_1 - \beta_2} + C_{11}\frac{\eta_1}{\eta_1 + \beta_3}(\frac{L}{U})^{\beta_3} + D_{11}\frac{\eta_1}{\eta_1 + \beta_4}(\frac{L}{U})^{\beta_4} &= e^{-\eta_1 z}, \\ A_{11}\frac{\eta_2}{\eta_2 + \beta_1}(\frac{L}{U})^{\beta_1} + B_{11}\frac{\eta_2}{\eta_2 + \beta_2}(\frac{L}{U})^{\beta_2} + C_{11}\frac{\eta_2}{\eta_2 - \beta_3} + D_{11}\frac{\eta_2}{\eta_2 - \beta_4} &= 0. \end{aligned}$$

We also know that

$$f(\beta_1) = f(\beta_2) = f(-\beta_3) = f(-\beta_4) = 0.$$

Accordingly, we have

$$-mv(x) + Lv(x) = 0 \quad \textit{for all } h < x < H.$$

A similar argument as before yields that

$$v(0) = E[e^{-m\tau_{(L,U)}}v(X_{\tau_{(L,U)}})1_{\{\tau_{(L,U)}<\infty\}}]$$

$$
\begin{aligned}
&= E[e^{-m\tau_{(L,U)}} 1_{\{X_{\tau_{(L,U)}} - H > z\}} 1_{\{\tau_{(L,U)} < \infty\}}] \\
&= E[e^{-m\tau_{(L,U)}} 1_{\{X_{\tau_{(L,U)}} - H > z\}}] \\
&= E[e^{-m\tau_U} 1_{\{X_{\tau_{(L,U)}} - H > z\}}],
\end{aligned}
$$

as $v(X_{\tau_{(L,U)}}) = 1_{\{X_{\tau_{(L,U)}} - H > z\}}$ on the set $\{\tau_{(L,U)} < \infty\}$, from which the result follows.

Corollary 3.3.1 *By the same parameters as Proposition 3.1, the following equations hold:*

$$
\begin{aligned}
E[e^{-m\tau_L} 1_{\{\tau_L < \tau_U\}}] &= \begin{bmatrix} A_2, & B_2, & C_2, & D_2 \end{bmatrix} \omega^T, && (3.13) \\
E[e^{-m\tau_L} 1_{\{X_{\tau_{(L,U)}} - h < z\}}] &= \begin{bmatrix} A_{21}, & B_{21}, & C_{21}, & D_{21} \end{bmatrix} \omega^T \quad \textit{for all } z \leq 0, && (3.14) \\
E[e^{-m\tau_L} 1_{\{X_{\tau_{(L,U)}} = h\}}] &= \begin{bmatrix} A_{22}, & B_{22}, & C_{22}, & D_{22} \end{bmatrix} \omega^T, && (3.15) \\
& && (3.16)
\end{aligned}
$$

where A_2, B_2, C_2, D_2 satisfy

$$
M \begin{bmatrix} A_2, & B_2, & C_2, & D_2 \end{bmatrix}^T = \begin{bmatrix} 0, & 1, & 0, & 1 \end{bmatrix}^T, \qquad (3.17)
$$

and A_{21}, B_{21}, C_{21}, D_{21} satisfy

$$
M \begin{bmatrix} A_{21}, & B_{21}, & C_{21}, & D_{21} \end{bmatrix}^T = \begin{bmatrix} 0, & 0, & 0, & e^{\eta_2 z} \end{bmatrix}^T, \qquad (3.18)
$$

and A_{22}, B_{22}, C_{22}, D_{22} satisfy

$$
A_2 = A_{21} + A_{22}, \; B_2 = B_{21} + B_{22}, \; C_2 = C_{21} + C_{22}, \; D_2 = D_{21} + D_{22}.
$$

Proof. It's very similar to prove (3.13). We define the function $v(x)$ of the form

$$
v(x) := \begin{cases} A_2 e^{-\beta_1(H-x)} + B_2 e^{-\beta_2(H-x)} + C_2 e^{\beta_3(h-x)} + D_2 e^{\beta_4(h-x)}, & h < x < H, \\ 0, & x \geq H, \\ 1, & x \leq h, \end{cases}
$$

where A_2, B_2, C_2 and D_2 are the solutions of (3.17). Again, $v(x)$ has the boundness and the continuity. Substituting the form of v and doing the integration in three regions, $\int_{-\infty}^{+\infty} = \int_{-\infty}^{h-x} + \int_{h-x}^{H-x} + \int_{H-x}^{+\infty}$, we have that, for all $h < x < H$,

$$\begin{aligned} -mv + Lv &= A_2 e^{-\beta_1(H-x)} f(\beta_1) + B_2 e^{-\beta_2(H-x)} f(\beta_2) + C_2 e^{\beta_3(h-x)} f(\beta_3) + D_2 e^{\beta_4(h-x)} f(\beta_4) - \\ &\quad \lambda p e^{-\eta_1(H-x)}[A_2 \frac{\eta_1}{\eta_1 - \beta_1} + B_2 \frac{\eta_1}{\eta_1 - \beta_2} + C_2 \frac{\eta_1}{\eta_1 + \beta_3} (\frac{L}{U})^{\beta_3} + D_2 \frac{\eta_1}{\eta_1 + \beta_4} (\frac{L}{U})^{\beta_4}] - \\ &\quad \lambda q e^{\eta_2(h-x)}[A_2 \frac{\eta_2}{\eta_2 + \beta_1} (\frac{L}{U})^{\beta_1} + B_2 \frac{\eta_2}{\eta_2 + \beta_2} (\frac{L}{U})^{\beta_2} + C_2 \frac{\eta_2}{\eta_2 - \beta_3} + D_2 \frac{\eta_2}{\eta_2 - \beta_4} - 1], \end{aligned}$$

where $f(\beta) := G(\beta) - m$. From (3.17) we know that

$$\begin{aligned} A_2 \frac{\eta_1}{\eta_1 - \beta_1} + B_2 \frac{\eta_1}{\eta_1 - \beta_2} + C_2 \frac{\eta_1}{\eta_1 + \beta_3} (\frac{L}{U})^{\beta_3} + D_2 \frac{\eta_1}{\eta_1 + \beta_4} (\frac{L}{U})^{\beta_4} &= 0, \\ A_2 \frac{\eta_2}{\eta_2 + \beta_1} (\frac{L}{U})^{\beta_1} + B_2 \frac{\eta_2}{\eta_2 + \beta_2} (\frac{L}{U})^{\beta_2} + C_2 \frac{\eta_2}{\eta_2 - \beta_3} + D_2 \frac{\eta_2}{\eta_2 - \beta_4} &= 1. \end{aligned}$$

We also know that

$$f(\beta_1) = f(\beta_2) = f(-\beta_3) = f(-\beta_4) = 0.$$

Accordingly, we have

$$-mv(x) + Lv(x) = 0 \quad \textit{for all } h < x < H.$$

A similar argument as before yields that

$$\begin{aligned} v(0) &= E[e^{-m\tau_{(L,U)}} v(X_{\tau_{(L,U)}}) 1_{\{\tau_{(L,U)} < \infty\}}] \\ &= E[e^{-m\tau_{(L,U)}} 1_{\{X_{\tau_{(L,U)}} \leq h\}} 1_{\{\tau_{(L,U)} < \infty\}}] \\ &= E[e^{-m\tau_{(L,U)}} 1_{\{X_{\tau_{(L,U)}} \leq h\}}] \\ &= E[e^{-m\tau_{(L,U)}} 1_{\{\tau_L < \tau_U\}}] \\ &= E[e^{-m\tau_L} 1_{\{\tau_L < \tau_U\}}], \end{aligned}$$

as $v(X_{\tau_{(L,U)}}) = 1_{\{\tau_L < \tau_U\}}$ on the set $\{\tau_{(L,U)} < \infty\}$, from which the result follows.

Now we prove (3.14) and (3.15). Again, by letting $z = 0$ and taking the difference of (3.13) and (3.14), we get (3.15) immediately. The proof of (3.14) is also similar. Define the function v to be

$$v(x) := \begin{cases} 1, & x < h + z, \\ 0, & h + z \le x < h, \\ A_{21}e^{-\beta_1(H-x)} + B_{21}e^{-\beta_2(H-x)} + C_{21}e^{\beta_3(h-x)} + D_{21}e^{\beta_4(h-x)}, & h \le x \le H, \\ 0, & x > H, \end{cases}$$

where A_{21}, B_{21}, C_{21} and D_{21} are the solutions of the equations (3.17). Substitute to obtain that, for all $h \le x \le H$,

$$\begin{aligned} -mv + Lv \;=\;& A_{21}e^{-\beta_1(H-x)}f(\beta_1) + B_{21}e^{-\beta_2(H-x)}f(\beta_2) + C_{21}e^{\beta_3(h-x)}f(\beta_3) + D_{21}e^{\beta_4(h-x)}f(\beta_4) - \\ & \lambda p e^{-\eta_1(H-x)}[A_{11}\frac{\eta_1}{\eta_1-\beta_1} + B_{11}\frac{\eta_1}{\eta_1-\beta_2} + C_{21}\frac{\eta_1}{\eta_1+\beta_3}(\frac{L}{U})^{\beta_3} + D_{21}\frac{\eta_1}{\eta_1+\beta_4}(\frac{L}{U})^{\beta_4}] - \\ & \lambda q e^{\eta_2(h-x)}[A_{21}\frac{\eta_2}{\eta_2+\beta_1}(\frac{L}{U})^{\beta_1} + B_{21}\frac{\eta_2}{\eta_2+\beta_2}(\frac{L}{U})^{\beta_2} + \\ & C_{21}\frac{\eta_2}{\eta_2-\beta_3} + D_{21}\frac{\eta_2}{\eta_2-\beta_4} - e^{\eta_2 z}]. \end{aligned}$$

From (3.10) we know that

$$\begin{aligned} A_{21}\frac{\eta_1}{\eta_1-\beta_1} + B_{21}\frac{\eta_1}{\eta_1-\beta_2} + C_{21}\frac{\eta_1}{\eta_1+\beta_3}(\frac{L}{U})^{\beta_3} + D_{21}\frac{\eta_1}{\eta_1+\beta_4}(\frac{L}{U})^{\beta_4} &= 0, \\ A_{21}\frac{\eta_2}{\eta_2+\beta_1}(\frac{L}{U})^{\beta_1} + B_{21}\frac{\eta_2}{\eta_2+\beta_2}(\frac{L}{U})^{\beta_2} + C_{21}\frac{\eta_2}{\eta_2-\beta_3} + D_{21}\frac{\eta_2}{\eta_2-\beta_4} &= e^{\eta_2 z}. \end{aligned}$$

We also know that

$$f(\beta_1) = f(\beta_2) = f(-\beta_3) = f(-\beta_4) = 0.$$

Accordingly, we have

$$-mv(x) + Lv(x) \;=\; 0 \quad \textit{for all } h < x < H.$$

A similar argument as before leads to

$$v(0) \;=\; E[e^{-m\tau_{(L,U)}}v(X_{\tau_{(L,U)}})1_{\{\tau_{(L,U)}<\infty\}}]$$

$$
\begin{aligned}
&= E[e^{-m\tau_{(L,U)}} 1_{\{X_{\tau_{(L,U)}}-h<z\}} 1_{\{\tau_{(L,U)}<\infty\}}] \\
&= E[e^{-m\tau_{(L,U)}} 1_{\{X_{\tau_{(L,U)}}-h<z\}}] \\
&= E[e^{-m\tau_L} 1_{\{X_{\tau_{(L,U)}}-h<z\}}],
\end{aligned}
$$

as $v(X_{\tau_{(L,U)}}) = 1_{\{X_{\tau_{(L,U)}}-h<z\}}$ on the set $\{\tau_{(L,U)} < \infty\}$, from which the result follows.

Corollary 3.3.2 *By the same parameters as Proposition 3.1, the following equation holds:*

$$
E[e^{-m\tau_{(L,U)}}] = \begin{bmatrix} A, & B, & C, & D \end{bmatrix} \omega^T, \tag{3.19}
$$

where $A = A_1 + A_2$, $B = B_1 + B_2$, $C = C_1 + C_2$, $D = D_1 + D_2$.

Proof: Since

$$
e^{-m\tau_{(L,U)}} = e^{-m\tau_U} 1_{\{\tau_U<\tau_L\}} + e^{-m\tau_L} 1_{\{\tau_L<\tau_U\}},
$$

(3.19) holds immediately.

Proposition 3.3 *Given the forms of linear systems, the solution to each system is unique.*

Proof. Given

$$
M = \begin{bmatrix}
1 & 1 & (\frac{L}{U})^{\beta_{3,m}} & (\frac{L}{U})^{\beta_{4,m}} \\
(\frac{L}{U})^{\beta_{1,m}} & (\frac{L}{U})^{\beta_{2,m}} & 1 & 1 \\
\frac{\eta_1}{\eta_1-\beta_{1,m}} & \frac{\eta_1}{\eta_1-\beta_{2,m}} & \frac{\eta_1}{\eta_1+\beta_{3,m}}(\frac{L}{U})^{\beta_{3,m}} & \frac{\eta_1}{\eta_1+\beta_{4,m}}(\frac{L}{U})^{\beta_{4,m}} \\
\frac{\eta_2}{\eta_2+\beta_{1,m}}(\frac{L}{U})^{\beta_{1,m}} & \frac{\eta_2}{\eta_2+\beta_{2,m}}(\frac{L}{U})^{\beta_{2,m}} & \frac{\eta_2}{\eta_2-\beta_{3,m}} & \frac{\eta_2}{\eta_2-\beta_{4,m}}
\end{bmatrix},
$$

we only need to prove that the matrix is invertible, i.e., $\det(M) \neq 0$.

For simplicity, we write $\beta_{i,m}$ as β_i, $i = 1, 2, 3, 4$, and write $\frac{L}{U}$ as x $(0 < x < 1)$.

Using the cofactor expansion on the first row of the matrix, we get

$$
\begin{aligned}
\det(M) &= \begin{bmatrix} x^{\beta_2} & 1 & 1 \\ \frac{\eta_1}{\eta_1-\beta_2} & \frac{\eta_1}{\eta_1+\beta_3}x^{\beta_3} & \frac{\eta_1}{\eta_1+\beta_4}x^{\beta_4} \\ \frac{\eta_2}{\eta_2+\beta_2}x^{\beta_2} & \frac{\eta_2}{\eta_2-\beta_3} & \frac{\eta_2}{\eta_2-\beta_4} \end{bmatrix} - \begin{bmatrix} x^{\beta_1} & 1 & 1 \\ \frac{\eta_1}{\eta_1-\beta_1} & \frac{\eta_1}{\eta_1+\beta_3}x^{\beta_3} & \frac{\eta_1}{\eta_1+\beta_4}x^{\beta_4} \\ \frac{\eta_2}{\eta_2+\beta_1}x^{\beta_1} & \frac{\eta_2}{\eta_2-\beta_3} & \frac{\eta_2}{\eta_2-\beta_4} \end{bmatrix} + \\
& x^{\beta_3}\begin{bmatrix} x^{\beta_1} & x^{\beta_2} & 1 \\ \frac{\eta_1}{\eta_1-\beta_1} & \frac{\eta_1}{\eta_1-\beta_2} & \frac{\eta_1}{\eta_1+\beta_4}x^{\beta_4} \\ \frac{\eta_2}{\eta_2+\beta_1}x^{\beta_1} & \frac{\eta_2}{\eta_2+\beta_2}x^{\beta_2} & \frac{\eta_2}{\eta_2-\beta_4} \end{bmatrix} - x^{\beta_4}\begin{bmatrix} x^{\beta_1} & x^{\beta_2} & 1 \\ \frac{\eta_1}{\eta_1-\beta_1} & \frac{\eta_1}{\eta_1-\beta_2} & \frac{\eta_1}{\eta_1+\beta_3}x^{\beta_3} \\ \frac{\eta_2}{\eta_2+\beta_1}x^{\beta_2} & \frac{\eta_2}{\eta_2+\beta_2}x^{\beta_2} & \frac{\eta_2}{\eta_2-\beta_3} \end{bmatrix}.
\end{aligned}
$$

Expanding all the 3×3 matrices above, the determinant of M should be

$$
\begin{aligned}
\det(M)(x) &= (x^{\beta_2} - x^{\beta_1})[\frac{\eta_1}{\eta_1+\beta_3}\cdot\frac{\eta_2}{\eta_2-\beta_4}x^{\beta_3} - \frac{\eta_1}{\eta_1+\beta_4}\cdot\frac{\eta_2}{\eta_2-\beta_3}x^{\beta_4}] \\
&-(\frac{\eta_1}{\eta_1-\beta_2} - \frac{\eta_1}{\eta_1-\beta_1})(\frac{\eta_2}{\eta_2-\beta_4} - \frac{\eta_2}{\eta_2-\beta_3}) \\
&+(\frac{\eta_2}{\eta_2+\beta_2}x^{\beta_2} - \frac{\eta_2}{\eta_2+\beta_1}x^{\beta_1})(\frac{\eta_1}{\eta_1+\beta_4}x^{\beta_4} - \frac{\eta_1}{\eta_1+\beta_3}x^{\beta_3}) \\
&+(x^{\beta_3} - x^{\beta_4})[\frac{\eta_1}{\eta_1-\beta_1}\cdot\frac{\eta_2}{\eta_2+\beta_2}x^{\beta_2} - \frac{\eta_1}{\eta_1-\beta_2}\cdot\frac{\eta_2}{\eta_2+\beta_1}x^{\beta_1}] \\
&-(\frac{\eta_1}{\eta_1+\beta_4}x^{\beta_3+\beta_4} - \frac{\eta_1}{\eta_1+\beta_3}x^{\beta_3+\beta_4})(\frac{\eta_2}{\eta_2+\beta_2}x^{\beta_1+\beta_2} - \frac{\eta_2}{\eta_2+\beta_1}x^{\beta_1+\beta_2}) \\
&+(\frac{\eta_2}{\eta_2-\beta_4}x^{\beta_3} - \frac{\eta_2}{\eta_2-\beta_3}x^{\beta_4})(\frac{\eta_1}{\eta_1-\beta_2}x^{\beta_1} - \frac{\eta_1}{\eta_1-\beta_1}x^{\beta_2}) \\
&= -\eta_1\eta_2[\frac{(\beta_2-\beta_1)(\beta_4-\beta_3)}{(\eta_1+\beta_3)(\eta_1+\beta_4)(\eta_2+\beta_1)(\eta_2+\beta_2)}x^{\beta_1+\beta_2+\beta_3+\beta_4} \\
&-\frac{(\beta_1+\beta_4)(\beta_2+\beta_3)}{(\eta_1-\beta_1)(\eta_1+\beta_4)(\eta_2+\beta_2)(\eta_2-\beta_3)}x^{\beta_2+\beta_4} \\
&+\frac{(\beta_1+\beta_3)(\beta_2+\beta_4)}{(\eta_1-\beta_1)(\eta_1+\beta_3)(\eta_2+\beta_2)(\eta_2-\beta_4)}x^{\beta_2+\beta_3} \\
&+\frac{(\beta_1+\beta_3)(\beta_2+\beta_4)}{(\eta_1-\beta_2)(\eta_1+\beta_4)(\eta_2+\beta_1)(\eta_2-\beta_3)}x^{\beta_1+\beta_4} \\
&-\frac{(\beta_1+\beta_3)(\beta_2+\beta_4)}{(\eta_1-\beta_2)(\eta_1+\beta_3)(\eta_2+\beta_1)(\eta_2-\beta_4)}x^{\beta_1+\beta_3} \\
&+\frac{(\beta_2-\beta_1)(\beta_4-\beta_3)}{(\eta_1-\beta_1)(\eta_1-\beta_2)(\eta_2-\beta_3)(\eta_2-\beta_4)}].
\end{aligned}
$$

Let

$$a_1 = \eta_1 - \beta_1, \quad a_2 = \eta_1 - \beta_2, \quad a_3 = \eta_1 + \beta_3, \quad a_4 = \eta_1 + \beta_4,$$

$$b_1 = \eta_2 + \beta_1, \quad b_2 = \eta_2 + \beta_2, \quad b_3 = \eta_2 - \beta_3, \quad b_4 = \eta_2 - \beta_4,$$

$$c_{ij} = \beta_i + \beta_j, \quad i < j,$$

$$d_1 = \beta_2 - \beta_1, \quad d_2 = \beta_4 - \beta_3.$$

Substitute all the expressions above into the determinant and get

$$\begin{aligned}\det(M)(x) &= -\eta_1\eta_2[\frac{d_1 d_2}{a_3 a_4 b_1 b_2} x^{\beta_1+\beta_2+\beta_3+\beta_4} - \frac{c_{14}c_{23}}{a_1 a_4 b_2 b_3} x^{\beta_2+\beta_4} + \frac{c_{13}c_{24}}{a_1 a_3 b_2 b_4} x^{\beta_2+\beta_3} \\ &\quad + \frac{c_{13}c_{24}}{a_2 a_4 b_1 b_3} x^{\beta_1+\beta_4} - \frac{c_{14}c_{23}}{a_2 a_3 b_1 b_4} x^{\beta_1+\beta_3} - \frac{d_1 d_2}{a_1 a_2 b_3 b_4}]\end{aligned}$$

Differentiate $\det(M)$ with respect to x and get

$$\begin{aligned}\frac{d\det(M)}{dx} &= -\frac{\eta_1\eta_2}{x}[(\beta_1+\beta_2+\beta_3+\beta_4)\frac{d_1 d_2}{a_3 a_4 b_1 b_2} x^{\beta_1+\beta_2+\beta_3+\beta_4} - \frac{c_{14}c_{23}c_{24}}{a_1 a_4 b_2 b_3} x^{\beta_2+\beta_4} \\ &\quad + \frac{c_{13}c_{24}c_{23}}{a_1 a_3 b_2 b_4} x^{\beta_2+\beta_3} + \frac{c_{13}c_{24}c_{14}}{a_2 a_4 b_1 b_3} x^{\beta_1+\beta_4} - \frac{c_{14}c_{23}c_{13}}{a_2 a_3 b_1 b_4} x^{\beta_1+\beta_3}]\end{aligned}$$

Since

$$0 < \beta_1 < \eta_1 < \beta_2 < \infty,$$

$$0 < \beta_3 < \eta_2 < \beta_4 < \infty,$$

the following inequalities hold

$$a_1 = \eta_1 - \beta_1 > 0, \quad a_2 = \eta_1 - \beta_2 < 0,$$

$$a_3 = \eta_1 + \beta_3 > 0, \quad a_4 = \eta_1 + \beta_4 > 0,$$

$$b_1 = \eta_2 + \beta_1 > 0, \quad b_2 = \eta_2 + \beta_2 > 0,$$

$$b_3 = \eta_2 - \beta_3 > 0, \quad b_4 = \eta_2 - \beta_4 < 0,$$

$$c_{ij} = \beta_i + \beta_j > 0, \quad i < j,$$

$$d_1 = \beta_2 - \beta_1 > 0, \quad d_2 = \beta_4 - \beta_3 > 0,$$

which give

$$(\beta_1+\beta_2+\beta_3+\beta_4)\frac{d_1 d_2}{a_3 a_4 b_1 b_2} x^{\beta_1+\beta_2+\beta_3+\beta_4} \quad < \quad 0,$$

$$\begin{aligned}
\frac{c_{14}c_{23}c_{24}}{a_1a_4b_2b_3}x^{\beta_2+\beta_4} &> 0, \\
\frac{c_{13}c_{24}c_{23}}{a_1a_3b_2b_4}x^{\beta_2+\beta_3} &< 0, \\
\frac{c_{13}c_{24}c_{14}}{a_2a_4b_1b_3}x^{\beta_1+\beta_4} &< 0, \\
\frac{c_{14}c_{23}c_{13}}{a_2a_3b_1b_4}x^{\beta_1+\beta_3} &> 0.
\end{aligned}$$

Therefore,

$$\frac{d\det(M)}{dx} > 0.$$

Since $0 < x < 1$ and

$$\begin{aligned}
\det(M)(1) &= \det\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ \frac{\eta_1}{\eta_1-\beta_1} & \frac{\eta_1}{\eta_1-\beta_2} & \frac{\eta_1}{\eta_1+\beta_3} & \frac{\eta_1}{\eta_1+\beta_4} \\ \frac{\eta_2}{\eta_2+\beta_1} & \frac{\eta_2}{\eta_2+\beta_2} & \frac{\eta_2}{\eta_2-\beta_3} & \frac{\eta_2}{\eta_2-\beta_4} \end{bmatrix} \\
&= 0,
\end{aligned}$$

$\det(M) < 0$ holds on the interval $x \in (0, 1)$, which implies that matrix M is invertible.

3.4 Pricing Double Barrier Option via Laplace Transform

For the double exponential jump diffusion model, Proposition 3.1 has shown that the memoryless property of the exponential distribution leads to (1) the conditional memoryless property of the jump overshoot; (2) the conditional independence of the overshoot and the first passage time given that the overshoot is bigger than 0. Moreover, with the analytical solutions for the Laplace transforms of the first passage time given in Proposition 3.2, we will price the option via Laplace transform in this section.

We will focus on pricing the double knock-out call option. The option can be written as

$$
\begin{aligned}
C(k,T) &= e^{-rT}E[(S(T)-K)^{+}1_{\{\tau_{(L,U)}>T\}}] \\
&= e^{-rT}E[(S(T)-e^{-k})1_{\{\tau_{(L,U)}>T\}}1_{\{k>-logS(T)\}}], \qquad (3.20)
\end{aligned}
$$

where $k=-log(K)$ the transformed strike.

Proposition 3.4 *For ξ and α such that $0<\xi<\eta_1-1$ and $\alpha>max(G(\xi+1)-r,0)$, the Laplace transform with respect to k and T of $C(k,T)$ is given by*

$$
\begin{aligned}
f(\xi,\alpha) &= \int_0^\infty\int_{-\infty}^\infty e^{-\xi k-\alpha T}C(k,T)dkdT \\
&= \frac{1}{\xi(\xi+1)}\frac{1}{r+\alpha-G(\xi+1)}\cdot \\
&\quad \{S^{\xi+1}-U^{\xi+1}[A(r+\alpha)\frac{\eta_1}{\eta_1-(\xi+1)}+B(r+\alpha)] \\
&\quad -L^{\xi+1}[C(r+\alpha)\frac{\eta_2}{\eta_2+(\xi+1)}+D(r+\alpha)]\}
\end{aligned}
$$

where

$$
\begin{aligned}
A(m) &:= E[e^{-m\tau_U}1_{\{X_{\tau_{(L,U)}}>H\}}] \\
&= A_{11}(\frac{S}{U})^{\beta_{1,m}}+B_{11}(\frac{S}{U})^{\beta_{2,m}}+C_{11}(\frac{L}{S})^{\beta_{3,m}}+D_{11}(\frac{L}{S})^{\beta_{4,m}} \\
B(m) &:= E[e^{-m\tau_U}1_{\{X_{\tau_{(L,U)}}=H\}}] \\
&= A_{12}(\frac{S}{U})^{\beta_{1,m}}+B_{12}(\frac{S}{U})^{\beta_{2,m}}+C_{12}(\frac{L}{S})^{\beta_{3,m}}+D_{12}(\frac{L}{S})^{\beta_{4,m}} \\
C(m) &:= E[e^{-m\tau_L}1_{\{X_{\tau_{(L,U)}}<h\}}] \\
&= A_{21}(\frac{S}{U})^{\beta_{1,m}}+B_{21}(\frac{S}{U})^{\beta_{2,m}}+C_{21}(\frac{L}{S})^{\beta_{3,m}}+D_{21}(\frac{L}{S})^{\beta_{4,m}} \\
D(m) &:= E[e^{-m\tau_L}1_{\{X_{\tau_{(L,U)}}=h\}}] \\
&= A_{22}(\frac{S}{U})^{\beta_{1,m}}+B_{22}(\frac{S}{U})^{\beta_{2,m}}+C_{22}(\frac{L}{S})^{\beta_{3,m}}+D_{22}(\frac{L}{S})^{\beta_{4,m}}
\end{aligned}
$$

Proof. It follows from (3.20) and the Fubini theorem that

$$
\begin{aligned}
f(\xi,\alpha) &= \int_0^\infty \int_{-\infty}^\infty e^{-\xi k-(r+\alpha)T} E[(S(T)-e^{-k})1_{\{\tau_{(L,U)}>T\}}1_{\{k>-logS(T)\}}]dkdT \\
&= E[\int_0^\infty e^{-(r+\alpha)T}1_{\{\tau_{(L,U)}>T\}}(\int_{-logS(T)}^\infty e^{-\xi k}(S(T)-e^{-k})dk)dT] \\
&= \frac{1}{\xi(\xi+1)}E[\int_0^\infty e^{-(r+\alpha)T}1_{\{\tau_{(L,U)}>T\}}S(T)^{\xi+1}dT] \\
&= \frac{1}{\xi(\xi+1)}E[\int_0^\infty e^{-(r+\alpha)T}(1-1_{\{\tau_U<T\}}1_{\{\tau_U<\tau_L\}}-1_{\{\tau_L<T\}}1_{\{\tau_L<\tau_U\}})S(T)^{\xi+1}dT] \\
&= \frac{1}{\xi(\xi+1)}\{E[\int_0^\infty e^{-(r+\alpha)T}S(T)^{\xi+1}dT] \\
&\quad -E[e^{-(r+\alpha)\tau_U}1_{\{\tau_U<\tau_L\}}\int_0^\infty e^{-(r+\alpha)t}S(t+\tau_U)^{\xi+1}dt] \\
&\quad -E[e^{-(r+\alpha)\tau_L}1_{\{\tau_L<\tau_U\}}\int_0^\infty e^{-(r+\alpha)t}S(t+\tau_L)^{\xi+1}dt]\}.
\end{aligned}
$$

Again, using Fubini theorem, we get

$$
\begin{aligned}
& E[\int_0^\infty e^{-(r+\alpha)T}S(T)^{\xi+1}dT] \\
&= \int_0^\infty e^{-(r+\alpha)T}E[S(T)^{\xi+1}]dT \\
&= \int_0^\infty e^{-(r+\alpha)T}E[(Se^{X(T)})^{\xi+1}]dT \\
&= S^{\xi+1}\int_0^\infty e^{-(r+\alpha)T}e^{G(\xi+1)T}dT \\
&= \frac{S^{\xi+1}}{\alpha+r-G(\xi+1)}
\end{aligned}
$$

The strong Markov property of X implies that

$$
\begin{aligned}
E[1_{\{\tau_U<\tau_L\}}\int_0^\infty e^{-(r+\alpha)t}S(t+\tau_U)^{\xi+1}dt|F_{\tau_U}] &= 1_{\{\tau_U<\tau_L\}}S(\tau_U)^{\xi+1}\int_0^\infty e^{-(r+\alpha)t}e^{G(\xi+1)t}dt \\
&= 1_{\{\tau_U<\tau_L\}}\frac{S(\tau_U)^{\xi+1}}{r+\alpha-G(\xi+1)}.
\end{aligned}
$$

Therefore,

$$
E[e^{-(r+\alpha)\tau_U}1_{\{\tau_U<\tau_L\}}\int_0^\infty e^{-(r+\alpha)t}S(t+\tau_U)^{\xi+1}dt]
$$